GEOMETRY OF SYMMETRIC DETERMINANTAL LOCI

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Abstract. We study algebro-geometric properties of determinantal loci of $(n+1)\times(n+1)$ symmetric matrices and also their double covers for even ranks. Their singularities, Fano indices and birational geometries are studied in general. The double covers of symmetric determinantal loci of rank four are studied with special interest by noting their relation to the Hilbert schemes of conics on Grassmannians.

1. Introduction

Throughout this paper, we work over $\mathbb{C}$, the complex number field, and we fix a vector space V of dimension $n+1$.

We define $\mathrm{S}_r \subset \mathbb{P}(\mathsf{S}^2V^*)$ to be the locus of quadrics in $\mathbb{P}(V)$ of rank at most r. Taking a basis of V, S_r is defined by $(r+1)\times(r+1)$ minors of the generic $(n+1)\times(n+1)$ symmetric matrix. We call S_r the *symmetric determinantal locus of rank at most* r. For example, $\mathrm{S}_1 = v_2(\mathbb{P}(V^*))$ with $v_2(\mathbb{P}(V^*))$ being the second Veronese variety of $\mathbb{P}(V^*)$ and $\mathrm{S}_{n+1} = \mathbb{P}(\mathsf{S}^2V^*)$. There is a natural stratification of $\mathbb{P}(\mathsf{S}^2V^*)$ by S_r:

$$v_2(\mathbb{P}(V^*)) = \mathrm{S}_1 \subset \mathrm{S}_2 \subset \cdots \subset \mathrm{S}_n \subset \mathrm{S}_{n+1} = \mathbb{P}(\mathsf{S}^2V^*).$$

We call a point of $\mathrm{S}_r \setminus \mathrm{S}_{r-1}$ *a rank r point*. Similarly we define the symmetric determinantal locus S^*_r in the dual projective space $\mathbb{P}(\mathsf{S}^2V)$. It is a well-known fact that the stratification of $\mathbb{P}(\mathsf{S}^2V^*)$ by S_r and that of $\mathbb{P}(\mathsf{S}^2V)$ by S^*_r are reversed under the projective duality.

Recently, classical projective duality is highlighted in the study of derived categories of coherent sheaves on projective varieties, where the duality is called *homological projective duality* (HPD) due to Kuznetsov [19]. HPD is a powerful framework to describe the derived category of a projective variety with its dual variety, and has been worked out in several interesting examples such as Pfaffian varieties (i.e., determinantal loci of anti-symmetric matrices) [20] and the second Veronese variety S^*_1 [22]. Interestingly, it is often the case that we have interesting pairs of Calabi-Yau manifolds associated to HPDs [2, 20]. In a series of papers [9]–[11], we have studied the case S^*_2 and S_4 for $n=4$ in detail, where a pair of smooth Calabi-Yau threefolds X and Y appears, respectively, as a linear section of S^*_2 and the double cover of the orthogonal linear section of S_4 branched along the set of rank 3 points. It has been shown in [11] that these X and Y are derived-equivalent, indicating that S^*_2 and the double cover T_4 of S_4 (called *double quintic symmetroids*) are HPD to each other. Also, for $n=3$, we have established in [14] the relations between the derived categories of a 2-dimensional linear section X of S^*_2 and the double cover Y of the orthogonal linear section of S_4 branched along the set of rank 2 or 3 points after the inspiring works [21] and [15]. In the latter case of $n=3$,

X is known as an Enriques surface of Reye congruence, while Y is known as an Artin-Mumford double solid.

The aim of the present paper is to put an algebro-geometric ground for our work [11]. Indeed this is an extended version of the first part of [12]. In a companion paper [13], we will study homological properties of S_2^* and T_4 for the cases $n = 3, 4$ based on the results of this paper. In this paper, we are concerned with the birational geometry of S_r for general n from the viewpoint of minimal model theory. In particular, for even r, we present a precise description of the double covers T_r of S_r branched along S_{r-1}. If $r \le n$, we show that S_r and T_r are $\mathbb{Q}$-factorial $\frac{(2n+3-r)r-2}{2}$-dimensional Fano varieties with Picard number one and Fano index $\frac{r(n+1)}{2}$ with only canonical singularities in Subsection 2.1.

As an interesting application of these general results, we will consider *orthogonal* linear sections of S^*_{n+2-r} and T_r, which entail a pair of Calabi-Yau varieties of the same dimensions. These Calabi-Yau varieties naturally generalize those studied in [11, 12, 13] for $n = 4$, and indicates that HPD holds for S^*_{n+2-r} and T_r (see Subsection 3.6).

Below is the summary of the birational geometry of the double covering T_4 of S_4 for genreal n which we establish in this paper. Note that a general point of S_4 corresponds to a quadric of rank four in $\mathbb{P}(V)$. It has two connected $\mathbb{P}^1$-families of $(n-2)$-planes which we identify with the respective conics in $\mathrm{G}(n-1, V)$. The double cover T_4 will be defined as the space which parametrizes the connected families of $(n-2)$-planes in quadrics, and will be described by making precise connection to the Hibert scheme of conics in $\mathrm{G}(n-1, V)$. In Section 4, we show the following:

Theorem 1.1. *Set $\mathscr{Y} := \mathrm{T}_4$ and denote by $\mathscr{Y}_0$ the Hilbert scheme of conics in $\mathrm{G}(n-1, V)$. Then there is a commutative diagram of birational maps as follow:*

$$\begin{tikzcd}
 & & \mathscr{Y}_0 \arrow[d] & & \\
\mathscr{Y}_3 \arrow[rr, dashed, "(anti\text{-})flip"] \arrow[dr] & & \widetilde{\mathscr{Y}} \arrow[dl] \arrow[drr, "\rho_{\widetilde{\mathscr{Y}}}"] & & \\
 & \overline{\mathscr{Y}}' & & & \mathscr{Y} := \mathrm{T}_4,
\end{tikzcd}$$

where

- $\mathscr{Y}_3 := \mathrm{G}(3, \wedge^2 \mathfrak{Q})$ *with the universal quotient bundle* $\mathfrak{Q}$ *of* $\mathrm{G}(n-3, V)$*,*
- $\overline{\mathscr{Y}}'$ *is the normalization of the subvariety* $\overline{\mathscr{Y}}$ *of* $\mathrm{G}(3, \wedge^{n-1} V)$ *parametrizing 3-planes annihilated by at least $n-3$ linearly independent vectors in V by the wedge product* (Propositions 4.8, 4.9)*,*
- $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$ *is a small contraction with non-trivial fibers being copies of* $\mathbb{P}^{n-3}$ (Proposition 4.11)*,*
- $\mathscr{Y}_3 \dashrightarrow \widetilde{\mathscr{Y}}$ *is the* (*anti-*) *flip for the small contraction* $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$ (Section 4.4)*,*
- $\widetilde{\mathscr{Y}} \to \overline{\mathscr{Y}}'$ *is a small contraction with non-trivial fibers being copies of* $\mathbb{P}^5$ (Proposition 4.15)*,*
- $\rho_{\widetilde{\mathscr{Y}}} \colon \widetilde{\mathscr{Y}} \to \mathscr{Y}$ *is an extremal divisorial contraction* (Proposition 4.22(2))*,*
- $\mathscr{Y}_0 \to \widetilde{\mathscr{Y}}$ *is the blow-up along a smooth subvariety* (Section 4.4)*.*

In the course of the proof, we give an explicit construction of the Hilbert scheme $\mathscr{Y}_0$ of conics in $\mathrm{G}(n-1,V)$ in Subsection 4.2. In Section 5, the contraction $\rho_{\widetilde{\mathscr{Y}}}: \widetilde{\mathscr{Y}} \to \mathscr{Y}$ is studied in detail. Let $F_{\widetilde{\mathscr{Y}}}$ be $\rho_{\widetilde{\mathscr{Y}}}$-exceptional divisor and $G_{\mathscr{Y}}$ be its image in $\mathscr{Y}$. We determine the biregular structure of $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ by introducing a natural double cover of $F_{\widetilde{\mathscr{Y}}}$. Flattening of the morphism $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ is constructed in Section 5. Despite its technical nature, the flat morphism plays crucial roles for our caluculations of the cohomologies of $\mathscr{Y}$ in [13].

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Notation: We will denote by V_i an i-dimensional vector subspace of V.

2. Basics for symmetric determinantal loci S_r

As introduced in the preceding section, we denote by $\mathrm{S}_r \subset \mathbb{P}(\mathsf{S}^2V^*)$ the locus of quadrics in $\mathbb{P}(V)$ of rank at most r.

2.1. Springer type resolution $\widetilde{\mathrm{S}}_r$ of S_r.

Let $\mathfrak{Q}$ be the universal quotient bundle of rank r on $\mathrm{G}(n+1-r,V)$ and define the following projective bundle over $\mathrm{G}(n+1-r,V)$:

$$\widetilde{\mathrm{S}}_r := \mathbb{P}(\mathsf{S}^2\mathfrak{Q}^*) \to \mathrm{G}(n+1-r,V). \tag{2.1}$$

When $r=n+1$, we consider this as the projective bundle over a point

$$\widetilde{\mathrm{S}}_{n+1} = \mathbb{P}(\mathsf{S}^2V^*) \to \mathrm{pt}$$

with $\widetilde{\mathrm{S}}_{n+1} = \mathrm{S}_{n+1}$. Considering the (dual of the) universal exact sequence, we see that there is a canonical injection $\mathfrak{Q}^* \hookrightarrow V^* \otimes \mathcal{O}$, which entails the injection $\mathsf{S}^2\mathfrak{Q}^* \hookrightarrow \mathsf{S}^2V^* \otimes \mathcal{O}$. With this injection, composed with the natural surjection $\mathbb{P}(\mathsf{S}^2V^* \otimes \mathcal{O}) \to \mathbb{P}(\mathsf{S}^2V^*)$, we have a morphism

$$\widetilde{\mathrm{S}}_r = \mathbb{P}(\mathsf{S}^2\mathfrak{Q}^*) \to \mathbb{P}(\mathsf{S}^2V^*). \tag{2.2}$$

By construction, the pull-back of $\mathcal{O}_{\mathbb{P}(\mathsf{S}^2V^*)}(1)$ to $\widetilde{\mathrm{S}}_r$ is the tautological divisor $\mathcal{O}_{\mathbb{P}(\mathsf{S}^2\mathfrak{Q}^*)}(1)$, which we denote by $M_{\widetilde{\mathrm{S}}_r}$.

Proposition 2.1.
(1) *The image of the morphism (2.2) coincides with* S_r. *The induced morphism* $p_{\widetilde{\mathrm{S}}_r}: \widetilde{\mathrm{S}}_r \to \mathrm{S}_r$ *is a resolution of* S_r.
(2) $\widetilde{\mathrm{S}}_r = \{([V_{n+1-r}],[Q]) \mid V_{n+1-r} \subset \mathrm{Sing}\, Q\} \subset \mathrm{G}(n+1-r,V) \times \mathbb{P}(\mathsf{S}^2V^*)$, *where* Q *is a quadric in* $\mathbb{P}(V)$.

Proof. (1) Since the fiber of $\mathfrak{Q}^*$ over a point $[V_{n+1-r}] \in \mathrm{G}(n+1-r,V)$ is $(V/V_{n+1-r})^*$, the fiber of the projective bundle $\widetilde{\mathrm{S}}_r \to \mathrm{G}(n+1-r,V)$ over $[V_{n+1-r}]$ is $\mathbb{P}(\mathsf{S}^2(V/V_{n+1-r})^*)$, which parameterizes quadrics in $\mathbb{P}(V/V_{n+1-r}) \simeq \mathbb{P}^{r-1}$. The morphism $\mathbb{P}(\mathsf{S}^2\mathfrak{Q}^*) \to \mathbb{P}(\mathsf{S}^2V^*)$ sends $\mathbb{P}(\mathsf{S}^2(V/V_{n+1-r})^*)$ into $\mathbb{P}(\mathsf{S}^2V^*)$. Then the image is identified with quadrics in $\mathbb{P}(V)$ which are singular at $[V_{n+1-r}]$, or equivalently, symmetric matrices whose kernels contain $[V_{n+1-r}]$. Therefore the image is S_r. The morphism $p_{\widetilde{\mathrm{S}}_r}: \widetilde{\mathrm{S}}_r \to \mathrm{S}_r$ is one to one over the locus of matrices of rank r in S_r, since a

symmetric matrix of rank r with the kernel V_{n+1-r} determines uniquely the corresponding quadric in $\mathbb{P}(V/V_{n+1-r})$. Hence $\widetilde{\mathrm{S}}_r$ is birational to S_r under $p_{\widetilde{\mathrm{S}}_r}$. Finally, $\widetilde{\mathrm{S}}_r$ is smooth since it is a projective bundle, and hence $p_{\widetilde{\mathrm{S}}_r}$ is a resolution of S_r.

The assertion (2) easily follows from the proof of (1). $\square$

Using the Springer type resolution $p_{\widetilde{\mathrm{S}}_r}$, we can derive several properties of S_r.

• **Dimension.** Since $\widetilde{\mathrm{S}}_r$ is a $\mathbb{P}^{\binom{r+1}{2}-1}$-bundle over $\mathrm{G}(n+1-r,V)$, it holds

$$\dim \mathrm{S}_r = \dim \widetilde{\mathrm{S}}_r = \frac{(r+1)r}{2} - 1 + r(n+1-r). \tag{2.3}$$

• **Canonical divisor.** Since $\widetilde{\mathrm{S}}_r = \mathbb{P}(\mathsf{S}^2\mathfrak{Q}^*)$ and $\det \mathsf{S}^2\mathfrak{Q} \simeq \mathcal{O}_{\mathrm{G}(n+1-r,V)}(r+1)$, we have

$$K_{\widetilde{\mathrm{S}}_r} = -\binom{r+1}{2} M_{\widetilde{\mathrm{S}}_r} - (n-r)L_{\widetilde{\mathrm{S}}_r}, \tag{2.4}$$

where $M_{\widetilde{\mathrm{S}}_r}$ is the tautological divisor of $\mathbb{P}(\mathsf{S}^2\mathfrak{Q}^*)$ and $L_{\widetilde{\mathrm{S}}_r}$ is the pull-back of $\mathcal{O}_{\mathrm{G}(n+1-r,V)}(1)$.

In the sequel in this subsection, we assume that $r \leq n$.

• **Exceptional divisor.** By Proposition 2.1 (2) and $\rho(\widetilde{\mathrm{S}}_r/\mathrm{S}_r) = 1$, the exceptional locus E_r of $p_{\widetilde{\mathrm{S}}_r}$ is a prime divisor and the induced map $E_r \to \mathrm{S}_{r-1}$ is a $\mathbb{P}^{n+1-r}$-bundle over $\mathrm{S}_{r-1} \setminus \mathrm{S}_{r-2}$. We have

$$E_r = rM_{\widetilde{\mathrm{S}}_r} - 2L_{\widetilde{\mathrm{S}}_r}. \tag{2.5}$$

Indeed, note that we may write $E_r = aM_{\widetilde{\mathrm{S}}_r} - bL_{\widetilde{\mathrm{S}}_r}$ with some integers a and b since $M_{\widetilde{\mathrm{S}}_r}$ and $L_{\widetilde{\mathrm{S}}_r}$ generate $\mathrm{Pic}\,\widetilde{\mathrm{S}}_r$. Let $\mathbb{P} \simeq \mathbb{P}^{n+1-r}$ be the fiber of $E_r \to \mathrm{S}_{r-1}$ over a point of $\mathrm{S}_{r-1} \setminus \mathrm{S}_{r-2}$. Then, by (2.4) and $M_{\widetilde{\mathrm{S}}_r}|_{\mathbb{P}} = 0$, we have $K_{\widetilde{\mathrm{S}}_r}|_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-(n-r))$. Therefore, using $K_{\mathbb{P}} = K_{E_r}|_{\mathbb{P}} = (K_{\widetilde{\mathrm{S}}_r} + E_r)|_{\mathbb{P}}$, we obtain $E_r|_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-2)$. Thus $b = 2$. We have $a = r$ since the restriction of E_r to a fiber $\mathbb{P}(\mathsf{S}^2(V/V_{n+1-r})^*)$ of $\widetilde{\mathrm{S}}_r \to \mathrm{G}(n+1-r,V)$ is the locus of singular quadrics in $\mathbb{P}(V/V_{n+1-r})$, and it is a degree r hypersurface in $\mathbb{P}(\mathsf{S}^2(V/V_{n+1-r})^*)$.

• **Generic Singularity.** By $E_r|_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-2)$, we see that

$$\mathrm{S}_r \text{ has } \frac{1}{2}(1^{n+2-r})\text{-singularities along } S_{r-1} \setminus S_{r-2}, \tag{2.6}$$

hence $\mathrm{Sing}\,\mathrm{S}_r = \mathrm{S}_{r-1}$.

• **Discrepancy and Fano index.** The two equalities (2.4) and (2.5) give the following presentation of $K_{\widetilde{\mathrm{S}}_r}$:

$$K_{\widetilde{\mathrm{S}}_r} =_{\mathbb{Q}} -\frac{r(n+1)}{2} M_{\widetilde{\mathrm{S}}_r} + \frac{n-r}{2} E_r. \tag{2.7}$$

The pushforward of (2.7) immediately gives

$$K_{\mathrm{S}_r} =_{\mathbb{Q}} -\frac{r(n+1)}{2} M_{\mathrm{S}_r}. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain

$$K_{\widetilde{\mathrm{S}}_r} =_{\mathbb{Q}} p^*_{\widetilde{\mathrm{S}}_r} K_{\mathrm{S}_r} + \frac{n-r}{2} E_r.$$

In particular, S_r has only terminal singularities if $n > r$, and canonical singularities if $n = r$. S_r is $\mathbb{Q}$-factorial since $\widetilde{\mathrm{S}}_r$ is smooth and $p_{\widetilde{\mathrm{S}}_r}$ is a divisorial contraction.

• **Gorenstein index.** K_{S_r} is Cartier in case $n-r$ is even. In case $n-r$ is odd, $2K_{\mathrm{S}_r}$ is Cartier while K_{S_r} is not.

Indeed, when $n-r$ is even, the integral divisor $K_{\widetilde{\mathrm{S}}_r} - \frac{n-r}{2}E_r$ is the pull-back of a Cartier divisor on S_r by the Kawamata-Shokurov base point free theorem. Then, in this case, the formulas (2.7) and (2.8) mean linear equivalences. In particular, K_{S_r} is Cartier. In case $n-r$ is odd, we see the assertion by a similar argument and (2.6).

2.2. **Double cover $\mathbf{T}_r$ of $\mathbf{S}_r$ with even r.** Throughout in this subsection, we suppose r is even. When r is even, due to the fact that a quadric of even rank contains two connected families of maximal linear subspaces in it, the determinantal locus S_r has a natural double cover. We describe below the double cover by formulating Springer type morphism.

Note that any quadric of rank at most r contains $(n-\frac{r}{2})$-planes. We will introduce the variety U_r which parameterizes pairs $([\Pi],[Q])$ of quadrics Q of rank at most r and $(n-\frac{r}{2})$-planes $\mathbb{P}(\Pi)$ such that $\mathbb{P}(\Pi)\subset Q$. To parametrize $(n-\frac{r}{2})$-planes in $\mathbb{P}(V)$, consider the Grassmannian $\mathrm{G}(n-\frac{r}{2}+1,V)$. Let

$$0 \to \mathcal{W}^*_{\frac{r}{2}} \to V^* \otimes \mathcal{O}_{\mathrm{G}(n-\frac{r}{2}+1,V)} \to \mathcal{U}^*_{n-\frac{r}{2}+1} \to 0 \tag{2.9}$$

be the dual of the universal exact sequence on $\mathrm{G}(n-\frac{r}{2}+1,V)$, where $\mathcal{W}_{\frac{r}{2}}$ is the universal quotient bundle of rank $\frac{r}{2}$ and $\mathcal{U}_{n-\frac{r}{2}+1}$ is the universal subbundle of rank $n-\frac{r}{2}+1$. For brevity, we often omit the subscripts writing them by $\mathcal{U}$ and $\mathcal{W}$. For an $(n-\frac{r}{2})$-plane $\mathbb{P}(\Pi)\subset\mathbb{P}(V)$, there exists a natural surjection $\mathsf{S}^2V^* \to \mathsf{S}^2H^0(\mathbb{P}(\Pi),\mathcal{O}_{\mathbb{P}(\Pi)}(1))$ such that the projectivization of the kernel consists of the quadrics containing $\mathbb{P}(\Pi)$. By relativizing this surjection over $\mathrm{G}(n-\frac{r}{2}+1,V)$, we obtain the following surjection: $\mathsf{S}^2V^*\otimes\mathcal{O}_{\mathrm{G}(n-\frac{r}{2}-1,V)} \to \mathsf{S}^2\mathcal{U}^*$. Let $\mathcal{E}^*$ be the kernel of this surjection, and consider the following exact sequence:

$$0 \to \mathcal{E}^* \to \mathsf{S}^2V^*\otimes\mathcal{O}_{\mathrm{G}(n-\frac{r}{2}+1,V)} \to \mathsf{S}^2\mathcal{U}^* \to 0. \tag{2.10}$$

Now we set $\mathrm{U}_r := \mathbb{P}(\mathcal{E}^*)$ and denote by ρ_{U_r} the projection $\mathrm{U}_r \to \mathrm{G}(n-\frac{r}{2}+1,V)$. By (2.10), U_r is contained in $\mathrm{G}(n-\frac{r}{2}+1,V)\times\mathbb{P}(\mathsf{S}^2V^*)$. Since the fiber of $\mathcal{E}^*$ over $[\Pi]$ parameterizes quadrics in $\mathbb{P}(V)$ containing $\mathbb{P}(\Pi)$, we have

$$\mathrm{U}_r = \{([\Pi],[Q]) \mid \mathbb{P}(\Pi)\subset Q\} \subset \mathrm{G}(n-\frac{r}{2}+1,V)\times\mathbb{P}(\mathsf{S}^2V^*).$$

Note that Q in $([\Pi],[Q])\in\mathrm{U}_r$ is a quadric of rank at most r since quadrics contain $(n-\frac{r}{2})$-planes only when their ranks are at most r. Hence the symmetric determinantal locus S_r is the image of the natural projection $\mathrm{U}_r\to\mathbb{P}(\mathsf{S}^2V^*)$. Now we let

$$\mathrm{U}_r \xrightarrow{\pi_{\mathrm{U}_r}} \mathrm{T}_r \xrightarrow{\rho_{\mathrm{T}_r}} \mathrm{S}_r$$

be the Stein factorization of $\mathrm{U}_r\to\mathrm{S}_r$. By (2.10), the tautological divisor of $\mathbb{P}(\mathcal{E}^*)\to\mathrm{G}(n-\frac{r}{2}+1,V)$ is nothing but the pull-back of a hyperplane section of S_r. We set

$$M_{\mathrm{U}_r} := \pi^*_{\mathrm{U}_r}\circ\rho^*_{\mathrm{T}_r}\mathcal{O}_{\mathrm{S}_r}(1).$$

We denote by $\mathrm{U}_{r[Q]}$ the fiber of $\mathrm{U}_r\to\mathrm{S}_r$ over a point $[Q]\in\mathrm{S}_r$.

Proposition 2.2. *For a quadric Q of rank r, the fiber $\mathrm{U}_{r[Q]}$ is the orthogonal Grassmannian $\mathrm{OG}(\frac{r}{2},r)$ which consists of two connected components.*

Proof. Quadric Q of even rank r induces a non-degenerate symmetric bilinear form q on the quotient V/V_{n+1-r}, where V_{n+1-r} is the $(n+1-r)$-dimensional vector space such that $[V_{n+1-r}]$ is the vertex of Q. Then $(n-\frac{r}{2})$-planes on Q naturally correspond to the maximal isotropic subspaces in V/V_{n-r+1} with respect to q, which are parameterized by the orthogonal Grassmannian $\mathrm{OG}(\frac{r}{2}, r)$. $\square$

Proposition 2.3. *The finite morphism* $\mathrm{T}_r \to \mathrm{S}_r$ *is of degree two and is branched along* S_{r-1}.

Proof. By Proposition 2.2, the degree of $\mathrm{T}_r \to \mathrm{S}_r$ is two since $\mathrm{U}_{r[Q]}$ has two connected components for a quadric Q of rank r. If a quadric Q has rank at most $r-1$, the family of $(n-\frac{r}{2})$-planes in Q is connected. Hence we have the assertion. $\square$

By this proposition, we see that T_r parameterizes connected families of $(n-\frac{r}{2})$-planes in quadrics of rank at most r in $\mathbb{P}(V)$ (cf. Fig.1).

Definition 2.4. We call T_r the *double symmetric determinantal locus* of rank at most r. We call a point of $\rho_{\mathrm{T}_r}^{-1}(\mathrm{S}_i \setminus \mathrm{S}_{i-1})$ *a rank i point* for $1 \le i \le r$.

T_r inherits good properties from S_r as follows:

Proposition 2.5. (1) *The Picard number of* U_r *is two and* $\pi_{\mathrm{U}_r}\colon \mathrm{U}_r \to \mathrm{T}_r$ *is a Mori fiber space. In particular,* T_r *is $\mathbb{Q}$-factorial and has Picard number one.*
(2) T_r *has only Gorenstein canonical singularities and* $\operatorname{Sing} \mathrm{T}_r$ *is contained in the inverse image of* S_{r-2}*. In particular,* $\dim \operatorname{Sing} \mathrm{T}_r$ *is smaller than* $\dim \operatorname{Sing} \mathrm{S}_r$ *in case* $r \le n$.
(3) T_r *is a Fano variety with*

$$K_{\mathrm{T}_r} = -\frac{r(n+1)}{2} M_{\mathrm{T}_r}, \tag{2.11}$$

where M_{T_r} *is the pull-back of* $\mathcal{O}_{\mathrm{S}_r}(1)$.

Proof. (1) The Picard number of U_r is two since U_r is a projective bundle over $\mathrm{G}(n-\frac{r}{2}+1, V)$. Therefore the Picard number of T_r is one since the relative Picard number of $\pi_{\mathrm{U}_r}\colon \mathrm{U}_r \to \mathrm{T}_r$ is one. π_{U_r} is a Mori fiber space since a general fiber of π_{U_r} is a Fano variety by Proposition 2.2. T_r is $\mathbb{Q}$-factorial by [17, Lemma 5-1-5].
(2) To show the claim (2), we will construct the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{U}}_r \arrow[r, "\pi_{\widetilde{\mathrm{U}}_r}"] \arrow[d, "p_{\widetilde{\mathrm{U}}_r}"'] & \widetilde{\mathrm{T}}_r \arrow[r, "\rho_{\widetilde{\mathrm{T}}_r}"] \arrow[d, "p_{\widetilde{\mathrm{T}}_r}"'] & \widetilde{\mathrm{S}}_r \arrow[d, "p_{\widetilde{\mathrm{S}}_r}"'] \\
\mathrm{U}_r \arrow[r, "\pi_{\mathrm{U}_r}"'] & \mathrm{T}_r \arrow[r, "\rho_{\mathrm{T}_r}"'] & \mathrm{S}_r.
\end{tikzcd} \tag{2.12}$$

• $\widetilde{\mathrm{U}}_r$ is defined in $\mathrm{G}(\frac{r}{2}, \mathfrak{Q}) \times_{\mathrm{G}(n+1-r,V)} \mathbb{P}(\mathsf{S}^2\mathfrak{Q}^*)$, in a similar way to U_r, by

$$\widetilde{\mathrm{U}}_r := \{([\Pi], [Q]; [V_{n+1-r}]) \mid \mathbb{P}(\Pi) \subset Q \subset \mathbb{P}(V/V_{n+1-r})\}.$$

• Then the projection to the second factor yields a morphism $\widetilde{\mathrm{U}}_r \to \widetilde{\mathrm{S}}_r$ and the morphism $\mathrm{G}(\frac{r}{2}, \mathfrak{Q}) \times_{\mathrm{G}(n+1-r,V)} \mathbb{P}(\mathsf{S}^2\mathfrak{Q}^*) \to \mathrm{G}(n+1-\frac{r}{2}, V) \times \mathbb{P}(\mathsf{S}^2V^*)$ induces a morphism $p_{\widetilde{\mathrm{U}}_r}\colon \widetilde{\mathrm{U}}_r \to \mathrm{U}_r$. It is easy to see that $p_{\widetilde{\mathrm{U}}_r}$ is a birational morphism.
• Let

$$\widetilde{\mathrm{U}}_r \xrightarrow{\pi_{\widetilde{\mathrm{U}}_r}} \widetilde{\mathrm{T}}_r \xrightarrow{\rho_{\widetilde{\mathrm{T}}_r}} \widetilde{\mathrm{S}}_r$$

be the Stein factorization of $\widetilde{\mathrm{U}}_r \to \widetilde{\mathrm{S}}_r$. By the definition of Stein factorization, we have $\pi_{\widetilde{\mathrm{U}}_r *}\mathcal{O}_{\widetilde{\mathrm{U}}_r} = \mathcal{O}_{\widetilde{\mathrm{T}}_r}$ and $\pi_{\mathrm{U}_r *}\mathcal{O}_{\mathrm{U}_r} = \mathcal{O}_{\mathrm{T}_r}$. Therefore, by

$$p_{\widetilde{\mathrm{S}}_r *}\rho_{\widetilde{\mathrm{T}}_r *}\mathcal{O}_{\widetilde{\mathrm{T}}_r} = p_{\widetilde{\mathrm{S}}_r *}\rho_{\widetilde{\mathrm{T}}_r *}\pi_{\widetilde{\mathrm{U}}_r *}\mathcal{O}_{\widetilde{\mathrm{U}}_r} = \rho_{\mathrm{T}_r *}\pi_{\mathrm{U}_r *}p_{\widetilde{\mathrm{U}}_r *}\mathcal{O}_{\widetilde{\mathrm{U}}_r} = \rho_{\mathrm{T}_r *}\mathcal{O}_{\mathrm{T}_r}, \tag{2.13}$$

we see that the Stein factorization of $p_{\widetilde{S}_r} \circ \rho_{\widetilde{T}_r}$ is $\widetilde{\mathrm{T}}_r \to \mathrm{T}_r \to \mathrm{S}_r$. We denote by $p_{\widetilde{\mathrm{T}}_r} \colon \widetilde{\mathrm{T}}_r \to \mathrm{T}_r$ the induced morphism.

Now we have completed the diagram (2.12). Similarly to the proof of Proposition 2.3, we see that the branch locus of $\rho_{\widetilde{T}_r} \colon \widetilde{\mathrm{T}}_r \to \widetilde{\mathrm{S}}_r$ is $p_{\widetilde{\mathrm{S}}_r}$-exceptional divisor E_r. Since $\widetilde{\mathrm{U}}_r \to \widetilde{\mathrm{T}}_r$ is a Mori fiber space, $\widetilde{\mathrm{T}}_r$ has only rational singularities by [5] and in particular is Cohen-Macaulay. Therefore $\rho_{\widetilde{\mathrm{T}}_r}$ is flat. First we treat the case where $r = n+1$. Then T_{n+1} is Gorenstein since it is the double cover of $\mathrm{S}_{n+1} = \mathbb{P}(\mathsf{S}^2 V^*)$ branched along the divisor S_n. Thus T_{n+1} has only canonical singularities by [5]. $\operatorname{Sing}\mathrm{T}_{n+1}$ is contained in the inverse image of $\operatorname{Sing}\mathrm{S}_n = \mathrm{S}_{n-1}$. Now we have verified the assertion (2) in case $r = n+1$. Let us assume that $r \leq n$. Then, by (2.5), it holds that

$$\rho^*_{\widetilde{\mathrm{T}}_r}(\frac{r}{2}M_{\widetilde{\mathrm{S}}_r} - L_{\widetilde{\mathrm{S}}_r}) \sim (\rho^*_{\widetilde{\mathrm{T}}_r}E_r)_{\mathrm{red}} \tag{2.14}$$

and $\rho_{\widetilde{\mathrm{T}}_r *}\mathcal{O}_{\widetilde{\mathrm{T}}_r} = \mathcal{O}_{\widetilde{\mathrm{S}}_r} \oplus \mathcal{O}_{\widetilde{\mathrm{S}}_r}(-\frac{r}{2}M_{\widetilde{\mathrm{S}}_r} + L_{\widetilde{\mathrm{S}}_r})$. By (2.13), we see that $\rho_{\mathrm{T}_r *}\mathcal{O}_{\mathrm{T}_r} = \mathcal{O}_{\mathrm{S}_r} \oplus \mathcal{O}_{\mathrm{S}_r}(-\frac{r}{2}M_{\mathrm{S}_r} + L_{\mathrm{S}_r})$ with $L_{\mathrm{S}_r} := \pi_{\widetilde{\mathrm{S}}_r *}L_{\widetilde{\mathrm{S}}_r}$ and

$$\mathrm{T}_r = \operatorname{Spec}_{\mathrm{S}_r}\big(\mathcal{O}_{\mathrm{S}_r} \oplus \mathcal{O}_{\mathrm{S}_r}(-\frac{r}{2}M_{\mathrm{S}_r} + L_{\mathrm{S}_r})\big). \tag{2.15}$$

Pushing (2.14) forward by $p_{\widetilde{\mathrm{T}}_r}$, we obtain

$$\rho^*_{\mathrm{T}_r}(\frac{r}{2}M_{\mathrm{S}_r} - L_{\mathrm{S}_r}) \sim 0. \tag{2.16}$$

In particular, $\rho^*_{\mathrm{T}_r}L_{\mathrm{S}_r}$ is Cartier since so is M_{S_r}. Therefore K_{T_r} is Cartier by (2.4) and the formula $K_{\mathrm{T}_r} = \rho^*_{\mathrm{T}_r}K_{\mathrm{S}_r}$. Namely, T_r is Gorenstein. To show that T_r has only canonical singularities, let $f \colon \widetilde{\mathrm{R}}_r \to \widetilde{\mathrm{T}}_r$ be a resolution. Then, by the ramification formula, we have $K_{\widetilde{\mathrm{R}}_r} \geq f^*\rho^*_{\widetilde{\mathrm{T}}_r}K_{\widetilde{\mathrm{S}}_r}$. Since S_r has only canonical singularities, we have $K_{\widetilde{\mathrm{S}}_r} \geq p^*_{\widetilde{\mathrm{S}}_r}K_{\mathrm{S}_r}$. Therefore

$$K_{\widetilde{\mathrm{R}}_r} \geq f^*\rho^*_{\widetilde{\mathrm{T}}_r}K_{\widetilde{\mathrm{S}}_r} \geq f^*\rho^*_{\widetilde{\mathrm{T}}_r}p^*_{\widetilde{\mathrm{S}}_r}K_{\mathrm{S}_r} = f^*p^*_{\widetilde{\mathrm{T}}_r}\rho^*_{\mathrm{T}_r}K_{\mathrm{S}_r} = f^*p^*_{\widetilde{\mathrm{T}}_r}K_{\mathrm{T}_r}.$$

This means that T_r has only canonical singularities.

By (2.6) and (2.15), we see that T_r is smooth at the inverse image of a rank $r-1$ point $s \in \mathrm{S}_r$ since L_{S_r} generates the divisor class group at s and then (2.15) coincides with punctured universal cover near s.

(3) If $r = n+1$, then the canonical divisor of T_r is given by

$$-\binom{n+2}{2}M_{\mathrm{T}_r} + \frac{n+1}{2}M_{\mathrm{T}_r} = -\frac{(n+1)^2}{2}M_{\mathrm{T}_r}$$

since the degree of the branch locus S_n is $n+1$. If $r \leq n$, then the assertion follows from $K_{\mathrm{T}_r} = \rho^*_{\mathrm{T}_r}K_{\mathrm{S}_r}$, (2.4) and (2.16). □

Remark 2.6. It is useful to consider that $\widetilde{\mathrm{T}}_r \to \widetilde{\mathrm{S}}_r$ as in the diagram (2.12) is the family over $\mathrm{G}(n+1-r, V)$ of the double cover $\mathrm{T}_r \to \mathrm{S}_r$ for r-dimensional vector spaces V/V_{n+1-r} with $[V_{n+1-r}] \in \mathrm{G}(n+1-r, V)$.

2.3. **Dual situations and orthogonal linear sections.** To consider projective duality for the symmetric determinantal loci in $\mathbb{P}(\mathsf{S}^2V^*)$, the symmetric determinantal loci in $\mathbb{P}(\mathsf{S}^2V)$ naturally appear. Recall that we denote by S_r^* the symmetric determinantal locus of rank at most r in $\mathbb{P}(\mathsf{S}^2V)$. Similarly to S_r, S_1^* is the second Veronese variety $v_2(\mathbb{P}(V))$ and S_r^* is the r-secant variety of S_1^*. Corresponding to our definitions $\mathrm{U}_r, \mathrm{T}_r$ and $\widetilde{\mathrm{S}_r}$ for S_r in $\mathbb{P}(\mathsf{S}^2V^*)$, we have similar definitions $\mathrm{U}_r^*, \mathrm{T}_r^*$ and $\widetilde{\mathrm{S}_r^*}$ for S_r^* in $\mathbb{P}(\mathsf{S}^2V)$.

For a linear subspace $L_{k+1} \subset \mathsf{S}^2V^*$ of dimension $k+1$, we say that $\mathrm{S}_r \cap \mathbb{P}(L_{k+1})$ is a *linear section* of S_r if $\mathrm{S}_r \cap \mathbb{P}(L_{k+1})$ is of codimension $\dim \mathsf{S}^2V^* - (k+1)$ in S_r. Linear sections of S_r^* is defined for linear subspaces in S^2V in a similar way.

Let $L_{k+1}^{\perp} \subset \mathsf{S}^2V$ be the linear subspace orthogonal to L_{k+1} with respect to the dual pairing. For a triple $(\mathrm{S}_r, \mathrm{S}_s^*, L_{k+1})$, we say that linear sections $\mathrm{S}_r \cap \mathbb{P}(L_{k+1})$ and $\mathrm{S}_s^* \cap \mathbb{P}(L_{k+1}^{\perp})$ are mutually *orthogonal*. By slight abuse of terminology, we also call the pull-back of a linear section of S_r by the double cover $\mathrm{T}_r \to \mathrm{S}_r$ *a linear section of* T_r.

3. Pairs of Calabi-Yau sections and plausible duality

In this paper, we adopt the following definition of Calabi-Yau variety and also Calabi-Yau manifold.

Definition 3.1. A normal projective variety X is called *a Calabi-Yau variety* if X has only Gorenstein canonical singularities, and its canonical divisor is trivial and $h^i(\mathcal{O}_X) = 0$ for $0 < i < \dim X$. If X is smooth, then X is called *a Calabi-Yau manifold*. A smooth Calabi-Yau threefold is abbreviated as a Calabi-Yau threefold.

3.1. Calabi-Yau linear section of S_r.

Proposition 3.2. *Assume that $n-r$ is even and $r < n+1$. Then a general linear section $\mathrm{S}_r^{\mathrm{CY}}$ of codimension $\frac{r(n+1)}{2}$ is a Calabi-Yau variety of dimension $\frac{r(n+2-r)}{2}-1$ with only terminal (resp. canonical) singularities if $r < n$ (resp. $r = n$). Moreover, a general $\mathrm{S}_r^{\mathrm{CY}}$ is smooth if and only if $r \leq 2$.*

Proof. $\mathrm{S}_r^{\mathrm{CY}}$ has trivial canonical divisor by (2.8) since K_{S_r} is Cartier in case $n-r$ is even. Since S_r has only terminal (resp. canonical) singularities in case $r < n$ (resp. $r = n$) and is a Fano variety as we saw in the subsection 2.1, it holds that $h^i(\mathcal{O}_{\mathrm{S}_r}) = 0$ for any $i > 0$ and $h^i(\mathcal{O}_{\mathrm{S}_r}(-jM_{\mathrm{S}_r})) = 0$ for any $i < \dim \mathrm{S}_r$ and $j > 0$ by the Kodaira-Kawamata-Viehweg vanishing theorem. Therefore we have $h^i(\mathcal{O}_{\mathrm{S}_r^{\mathrm{CY}}}) = 0$ for any $0 < i < \dim \mathrm{S}_r^{\mathrm{CY}}$ by the Koszul complex. By a version of the Bertini theorem (cf. [1, Prop. 0.8]), a general $\mathrm{S}_r^{\mathrm{CY}}$ has only terminal (resp. canonical) singularities in case $r < n$ (resp. $r = n$). Therefore a general $\mathrm{S}_r^{\mathrm{CY}}$ is a Calabi-Yau variety.

Since $r < n+1$, $\operatorname{Sing} \mathrm{S}_r = \mathrm{S}_{r-1}$. Thus the second assertion is equivalent to that $\dim \mathrm{S}_{r-1} = \frac{r(r-1)}{2} - 1 + (r-1)(n+2-r) < \frac{r(n+1)}{2}$ holds if and only if $r \leq 2$. A proof of this claim is elementary. □

Remark 3.3. In case $n-r$ is odd, we can show the following by the same argument as in the proof of Proposition 3.2:

Linear sections of S_r of codimension $\frac{r(n+1)}{2}$ does not have trivial canonical divisors but bi-canonical divisors are trivial. Except this, the same properties as $\mathrm{S}_r^{\mathrm{CY}}$ hold for them.

By the above proposition, we observe that

$$\dim \mathrm{S}_r^{\mathrm{CY}} = \dim \mathrm{S}_{n+2-r}^{\mathrm{CY}} = \dim \mathrm{S}_{n+2-r}^{*\mathrm{CY}}. \tag{3.1}$$

This indicates certain duality between S_r and S^*_{n+2-r}. We will discuss this duality in Subsection 3.6.

If $r = 1$, then S_1 is isomorphic to the second Veronese variety $v_2(\mathbb{P}(V))$. Therefore its linear sections are complete intersections of quadrics in $\mathbb{P}(V)$.

In the next subsection, we adopt the dual setting and consider S^*_2 and its linear sections $\mathrm{S}_2^{*\mathrm{CY}}$ in detail.

3.2. **Rank two case and Calabi-Yau manifold X of a Reye congruence.** Consider the determinant locus S^*_2 in $\mathbb{P}(\mathsf{S}^2V)$ and also $\mathrm{U}^*_2, \mathrm{T}^*_2, \widetilde{\mathrm{S}^*_2}$ defined in the same way as $\mathrm{U}_2, \mathrm{T}_2, \widetilde{\mathrm{S}_2}$ for S_2 in $\mathbb{P}(\mathsf{S}^2V^*)$. Note that $\mathrm{U}^*_2 \simeq \mathrm{T}^*_2$ holds in this case.

Let us write the exact sequence (2.10) for S^*_2 by noting that $\mathrm{G}(n, V^*) = \mathbb{P}(V)$ and $\mathcal{U} = \Omega^1_{\mathbb{P}(V)}$:

$$0 \to \mathcal{E}^* \to \mathsf{S}^2V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathsf{S}^2T_{\mathbb{P}(V)}(-1) \to 0. \tag{3.2}$$

Proposition 3.4. $\mathcal{E} \simeq V^* \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$.

Proof. Taking fibers of (3.2) at a point $[V_1] \in \mathbb{P}(V)$, we obtain the exact sequence $0 \to V \otimes V_1 \to \mathsf{S}^2V \to \mathsf{S}^2(V/V_1) \to 0$. Therefore the fiber of $\mathcal{E}^*$ at $[V_1]$ is $V \otimes V_1$, which show the claim. $\square$

Therefore it holds that

$$\mathrm{T}^*_2 \simeq \mathrm{U}^*_2 := \mathbb{P}(\mathcal{E}^*) \simeq \mathbb{P}(V) \times \mathbb{P}(V).$$

Moreover, by the proof of Proposition 3.4, we see that the map $\mathrm{T}^*_2 \to \mathbb{P}(\mathsf{S}^2V)$ is given by $\mathbb{P}(V)\times\mathbb{P}(V) \ni ([\mathbf{v}], [\mathbf{w}]) \mapsto [\mathbf{v}\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{v}] \in \mathbb{P}(\mathsf{S}^2V)$. Therefore S^*_2, which is the image of this map, is nothing but the symmetric product $\mathsf{S}^2\mathbb{P}(V)$. In [12], we show that, by identifying $\mathsf{S}^2\mathbb{P}(V)$ with the Chow variety of degree two 0-cycles in $\mathbb{P}(V)$ (cf. [7]), $\widetilde{\mathrm{S}^*_2}$ is isomorphic to the Hilbert scheme of length two subschemes in $\mathbb{P}(V)$, and the Springer resolution $\widetilde{\mathrm{S}^*_2} \to \mathrm{S}^*_2$ coincides with the Hilbert-Chow morphism.

For brevity of notation, we fix the following definitions in what follows:

$$\mathscr{X} := \mathrm{S}^*_2 \ \text{ and } \ X := \text{a codimension } n+1 \text{ linear section of } \mathrm{S}^*_2.$$

In [24] (see also [12]), a general X is called a *Reye congruence* since it is isomorphic to a $(n-1)$-dimensional subvariety of $\mathrm{G}(2, V)$. By Proposition 3.2 and Remark 3.3, Reye congruence X is a Calabi-Yau variety when n is even; when n is odd, X has similar properties except that $2K_X \sim 0$. In particular, when $n = 3$, X is an Enriques surface (see [4]).

The proof of the following proposition is standard, so we omit it here (cf. [12]).

Proposition 3.5. *For a general X, it holds that*

$$\pi_1(X) \simeq \mathbb{Z}_2, \ \ \operatorname{Pic} X \simeq \mathbb{Z} \oplus \mathbb{Z}_2,$$

where the free part of $\operatorname{Pic} X$ *is generated by the class D of a hyperplane section of $\mathscr{X}$ restricted to X.*

When $n = 4$, X is a Calabi-Yau threefold with the following invariants [9, Proposition 2.1]:

$$\deg(X) = 35, \ \ c_2.D = 50, \ \ h^{2,1}(X) = 26, \ h^{1,1}(X) = 1,$$

where c_2 is the second Chern class of X.

3.3. **Calabi-Yau linear section of T_r.** In this subsection, we assume that r is even.

Proposition 3.6. *A general linear section $\mathrm{T}_r^{\mathrm{CY}}$ of codimension $\frac{r(n+1)}{2}$ is a Calabi-Yau variety of dimension $\frac{r(n+2-r)}{2} - 1$ with only canonical singularities. Moreover, a general $\mathrm{T}_r^{\mathrm{CY}}$ is smooth if $r \leq 4$.*

Proof. By (2.11), $\mathrm{T}_r^{\mathrm{CY}}$ has trivial canonical divisor. Since T_r is a Fano variety with only canonical singularities by Proposition 2.5, We can show that $h^i(\mathcal{O}_{\mathrm{T}_r^{\mathrm{CY}}}) = 0$ for any $0 < i < \dim \mathrm{T}_r^{\mathrm{CY}}$, and a general $\mathrm{T}_r^{\mathrm{CY}}$ has only canonical singularities in the same way as in the proof of Proposition 3.2. Therefore a general $\mathrm{T}_r^{\mathrm{CY}}$ is a Calabi-Yau variety.

Since $\mathrm{Sing}\,\mathrm{T}_r$ is contained in the inverse image of S_{r-2} by Proposition 2.5 (2), the second assertion follows once we show that $\dim \mathrm{S}_{r-2} = \frac{(r-1)(r-2)}{2} - 1 + (r-2)(n+3-r) < \frac{r(n+1)}{2}$ holds if and only if $r \leq 4$. A proof of the latter is elementary. □

We have already studied $\mathrm{T}_2^{\mathrm{CY}}$ in the subsection 3.2. We deal with $\mathrm{T}_4^{\mathrm{CY}}$ in detail in the subsection 3.4.

3.4. **Rank four case and Calabi-Yau manifold Y.** For brevity of notation, we introduce the following definitions:

$$\mathscr{H} := \mathrm{S}_4, \quad \mathscr{U} := \widetilde{\mathrm{S}}_4, \quad \mathscr{Y} := \mathrm{T}_4, \quad \mathscr{Z} := \mathrm{U}_4,$$

while retaining the notation $\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3 \subset \mathscr{H}$. We denote by $\mathscr{Z}_{[Q]}$ the fiber of the morphism $\mathscr{Z} \to \mathscr{H}$ over a point $[Q]$. Recall that $\pi_{\mathrm{U}_4} = \pi_{\mathscr{Z}} : \mathscr{Z} \to \mathscr{Y}$ is defined by the Stein factorization $\mathscr{Z} \to \mathscr{Y} \to \mathscr{H}$ of $\mathscr{Z} \to \mathscr{H}$.

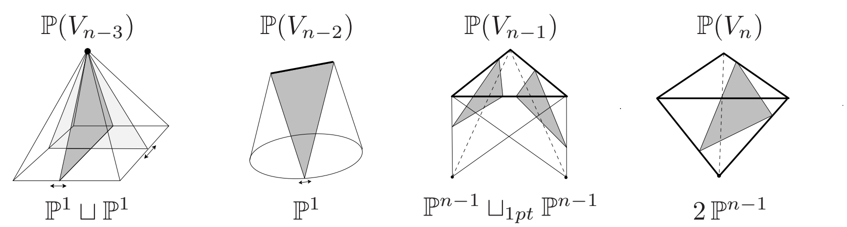

Fig.1. Quadrics Q of rank at most four in $\mathbb{P}(V)$ and families of $(n-2)$-planes therein. The singular loci of Q are written by $\mathbb{P}(V_k)$ with $k = n + 1 - \mathrm{rk}\,\mathrm{Q}$. Also the parameter spaces of the planes in each Q are shown ($\mathbb{P}^{n-1} \sqcup_{1pt} \mathbb{P}^{n-1}$ represents the union of $\mathbb{P}^{n-1}$'s intersecting at one point). See also Fig.2 in the subsection 4.7.

Proposition 3.7. *If* $\operatorname{rank} Q = 4$*, then* $\mathscr{X}_{[Q]}$ *is a disjoint union of two smooth rational curves, each of which is identified with a conic in* $\mathrm{G}(n-1, V)$*. If* $\operatorname{rank} Q = 3$*, then* $\mathscr{X}_{[Q]}$ *is a smooth rational curve, which is also identified with a conic in* $\mathrm{G}(n-1, V)$*. If* $\operatorname{rank} Q = 2$*, then* $\mathscr{X}_{[Q]}$ *is the union of two* $\mathbb{P}^{n-1}$*'s intersecting at one point. If* $\operatorname{rank} Q = 1$*, then* $\mathscr{X}_{[Q]}$ *is a (non-reduced)* $\mathbb{P}^{n-1}$*. In particular,* $\pi_{\mathscr{X}} \colon \mathscr{X} \to \mathscr{Y}$ *is generically a conic bundle.*

Proof. If $\operatorname{rank} Q = 4$, the fiber $\mathscr{X}_{[Q]}$ consists of two disconnected components, and is isomorphic to the orthogonal Grassmannian $\mathrm{OG}(2, 4)$ by Proposition 2.2. To be more explicit, let $\mathbb{P}(V_{n-3}) \subset \mathbb{P}(V)$ be the vertex of Q. Then the quadric Q is the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ with the vertex $\mathbb{P}(V_{n-3})$. There are two distinct $\mathbb{P}^1$-families of lines in $\mathbb{P}^1 \times \mathbb{P}^1$. Each of the families can be understood as the corresponding conic in $\mathrm{G}(2, V/V_{n-3})$, which gives one of the connected components of $\mathrm{OG}(2, 4)$. Under the natural map $\mathrm{G}(2, V/V_{n-3}) \to \mathrm{G}(n-1, V)$, we have two $\mathbb{P}^1$- families of 2-planes in Q parameterized by the conics in $\mathrm{G}(n-1, V)$.

If $\operatorname{rank} Q = 3$, the vertex of the quadric Q is a $\mathbb{P}(V_{n-2}) \subset \mathbb{P}(V)$. The quadric Q is the cone over a conic with the vertex $\mathbb{P}(V_{n-2})$. The conic is contained in $\mathbb{P}(V/V_{n-2}) = \mathrm{G}(1, V/V_{n-2})$, and can be identified with a conic in $\mathrm{G}(n-1, V)$ under the natural map $\mathrm{G}(1, V/V_{n-2}) \to \mathrm{G}(n-1, V)$.

If $\operatorname{rank} Q = 2$, then the quadric Q has a vertex $\mathbb{P}(V_{n-1}) \subset \mathbb{P}(V)$ and is the union of two $(n-1)$-planes intersecting along the $(n-2)$-plane $\mathbb{P}(V_{n-1})$. Hence $\mathscr{X}_{[Q]} \subset \mathrm{G}(n-1, V)$ is given by the union of the corresponding $\mathbb{P}^{n-1}$'s, i.e., $\mathrm{G}(n-1, n)$'s in $\mathrm{G}(n-1, V)$, which intersect at one point $\mathbb{P}(V_{n-1})$.

If $\operatorname{rank} Q = 1$, then Q is a double $(n-1)$-plane. Thus $\mathscr{X}_{[Q]}$ is a (non-reduced) $\mathbb{P}^{n-1} \cong \mathrm{G}(n-1, n)$. □

We write by $G^1_{\mathscr{Y}}$ (resp. $G^2_{\mathscr{Y}}$, $G_{\mathscr{Y}}$) the inverse image under $\rho_{\mathscr{Y}}$ of S_1 (resp. $\mathrm{S}_2 \setminus \mathrm{S}_1$, S_2). We note that $G_{\mathscr{Y}} \simeq \mathrm{S}_1 \simeq \mathsf{S}^2\mathbb{P}(V^*)$ and $G^1_{\mathscr{Y}} \simeq \mathrm{S}_2 \simeq v_2(\mathbb{P}(V^*))$ since S_2 is contained in the branch locus of $\rho_{\mathscr{Y}}$. Using these, we summarize our construction above in the following diagram:

$$
\begin{tikzcd}
 & & \mathscr{X} \arrow[r, "\text{proj. bundle}", "\rho_{\mathscr{X}}"'] \arrow[d, "\pi_{\mathscr{X}}"] & \mathrm{G}(n-1, V) \\
G^1_{\mathscr{Y}} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\wr"] & G_{\mathscr{Y}} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\wr"] & \mathscr{Y} \arrow[d, "\rho_{\mathscr{Y}}"] & \\
v_2(\mathbb{P}(V^*)) \subset & \mathsf{S}^2\mathbb{P}(V^*) \subset & \mathscr{H}, &
\end{tikzcd}
\tag{3.3}
$$

where $\pi_{\mathscr{X}}$ is a $\mathbb{P}^1$-fibration over $\mathscr{Y} \setminus G_{\mathscr{Y}}$ by Proposition 3.7. In Section 4, we will construct a nice desingularization $\widetilde{\mathscr{Y}}$ of $\mathscr{Y}$. Also, in Sections 4 and 5, we will study the geometry of $\widetilde{\mathscr{Y}} \to \mathscr{Y}$ along the loci $G_{\mathscr{Y}}$ and $G^1_{\mathscr{Y}}$ in full detail.

Now consider the linear section of $\mathscr{Y} = T_4$ and we set

$$Y := \mathrm{T}_4^{\mathrm{CY}}.$$

By Proposition 3.6, a general Y is a Calabi-Yau manifold of dimension $2n-5$.

By using the fibration $\pi_{\mathscr{X}} \colon \mathscr{X} \to \mathscr{Y}$, it is possible to compute several invariants of Y. Computations have been done for the case $n = 4$ in [9, Prop.3.11 and Prop.3.12], [12], which we summarize below:

Proposition 3.8. *A general Y is a simply connected smooth Calabi-Yau 3-fold such that* $\operatorname{Pic} Y = \mathbb{Z}[M]$, $M^3 = 10$, $c_2(Y).M = 40$ *and* $e(Y) = -50$. *In particular,* $h^{1,1}(Y) = 1$ *and* $h^{1,2}(Y) = 26$.

It should be noted here that the Spec construction (2.15) of $\mathrm{T}_4 = \mathscr{Y}$ generalizes the covering constructed in [9, eq.(3.4)] for $n = 4$.

In the following two subsections, we discuss two plausible dualities between S^*_a and T_b for certain pairs of a and b.

3.5. **Linear duality and beyond.** The exact sequence (2.10) means that the fibers of $\mathsf{S}^2\mathcal{U}$ and $\mathcal{E}^*$ over a point of $\mathrm{G}(n+1-\frac{r}{2}, V)$ are the orthogonal spaces to each other when we consider them as subspaces in $\mathsf{S}^2 V$ and $\mathsf{S}^2 V^*$, respectively. The pair $\mathsf{S}^2\mathcal{U}$ and $\mathcal{E}^*$ is an example of *orthogonal bundles*.

In [19, §8], Kuznetsov has established the homological projective duality between a projective bundle $\mathbb{P}(\mathcal{V})$ over a smooth base S and its orthogonal bundle $\mathbb{P}(\mathcal{V}^{\perp})$ for a globally generated vector bundle $\mathcal{V}$ on S. He has called this duality *linear duality* in [23]. Due to this general result, we know that $\mathbb{P}(\mathsf{S}^2\mathcal{U})$ and $\mathbb{P}(\mathcal{E}^*)$ are homological projective dual. Note that $\mathbb{P}(\mathsf{S}^2\mathcal{U}) = \widetilde{\mathrm{S}^*}_{n+1-\frac{r}{2}}$ and $\mathbb{P}(\mathcal{E}^*) = \mathrm{U}_r$. Mutually orthogonal linear sections X and Z of $\mathbb{P}(\mathsf{S}^2\mathcal{U})$ and $\mathbb{P}(\mathcal{E}^*)$ of codimensions $\operatorname{rank} \mathsf{S}^2\mathcal{U}$ and $\operatorname{rank} \mathcal{E}^*$ respectively have the equal dimensions, $\dim \mathrm{G}(n+1-\frac{r}{2}, V) - 1 = \frac{r}{2}(n+1-\frac{r}{2}) - 1$, and are derived equivalent by [19, §8]. Let Y be the double cover of the image of Z on $\mathbb{P}(\mathsf{S}^2 V^*)$. The derived equivalence between X and Z indicates that there is some relationship between non-commutative resolutions of $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$. Indeed, in [14], we have shown that this is the case when $n = 3$ and $r = 4$. Note that in this case, a general X is a so-called Enriques-Fano threefold and a general Y is a del Pezzo surface of degree two [ibid.]. In this case (of $n = 3$ and $r = 4$), we can also investigate the derived categories of mutually orthogonal linear sections of S^*_2 and T_4 for a triple $(\mathrm{S}_4, \mathrm{S}^*_2, L_4)$, which define, respectively, an Enriques surface of Reye congruence and Artin Mumford double solid. In [13], we have found natural Lefschetz collections, which indicates that certain non-commutative resolutions of S^*_2 and T_4 are homological projective dual to each other. One may suspect that, with finding suitable Lefschetz collections, non-commutative resolutions of $\mathrm{S}^*_{n+1-\frac{r}{2}}$ and T_r are homologically projective dual to each other in general.

3.6. **Plausible duality.** Assume that r is even. Then $n-(n+2-r)$ is also even. Therefore we obtain mutually orthogonal Calabi-Yau linear sections $\mathrm{S}^{*\mathrm{CY}}_{n+2-r}$ and $\mathrm{T}^{\mathrm{CY}}_r$ by Propositions 3.2 and 3.6.

We suspect an equivalence of the derived categories of certain non-commutative resolutions of orthogonal linear sections $\mathrm{S}^{*\mathrm{CY}}_{n+2-r}$ and $\mathrm{T}^{\mathrm{CY}}_r$ rather than $\mathrm{S}^{*\mathrm{CY}}_{n+2-r}$ and $\mathrm{S}^{\mathrm{CY}}_r$. More generally, we speculate that certain non-commutative resolutions of S^*_{n+2-r} and T_r with suitable Lefschetz collections for each are homologically projective dual. In fact, this is established in case $r = n+1$ [22] (called Veronese-Clifford duality). Note that in case $n = r = 4$, both $\mathrm{S}^{*\mathrm{CY}}_2 = X$ and $\mathrm{T}^{\mathrm{CY}}_4 = Y$ are smooth, and hence they are of considerable interest. In [13], we have constructed (dual) Lefschetz collections in the derived categories of $\widetilde{\mathrm{S}}^*_2$ and T_4, and have proved the derived equivalence between $\mathrm{S}^{*\mathrm{CY}}_2$ and $\mathrm{T}^{\mathrm{CY}}_4$ in [11] using the properties of these collections.

Having these applications in mind, in the rest of this paper, we study the birational geometry of $\mathscr{Y} = \mathrm{T}_4$ for general n. Since we will be concentrated on the case $r = 4$, we will extensively use the notation introduced in the beginning of the subsection 3.4.

4. Birational geometry of $\mathscr{Y}$

Proposition 3.7 indicates a correspondence between points in $\mathscr{Y}$ and conics in $\mathrm{G}(n-1, V)$. In this section, we explicitly construct a birational map between $\mathscr{Y}$ and the Hilbert scheme $\mathscr{Y}_0$ of conics in $\mathrm{G}(n-1, V)$.

4.1. **Conics and planes in** $\mathrm{G}(n-1, V)$**.** Let q be a conic in $\mathrm{G}(n-1, V)$ and $\mathbb{P}_q$ the plane spanned by q. Noting that $\mathrm{G}(n-1, V)$ is the intersection of the Plücker quadrics in $\mathbb{P}(\wedge^{n-1}V)$, we see that either $\mathbb{P}_q \subset \mathrm{G}(n-1, V)$ or $\mathrm{G}(n-1, V) \cap \mathbb{P}_q = q$ holds for $\mathbb{P}_q$.

When $\mathbb{P}_q \subset \mathrm{G}(n-1, V)$, we note that there are exactly two types of planes contained in $\mathrm{G}(n-1, V) \subset \mathbb{P}(\wedge^{n-1}V)$:

$$
\begin{aligned}
\mathrm{P}_{V_{n-2}} &:= \{[\Pi] \in \mathrm{G}(n-1, V) \mid V_{n-2} \subset \Pi\} \cong \mathbb{P}^2 \qquad (\rho\text{-plane}),\\
\mathrm{P}_{V_{n-3}V_n} &:= \{[\Pi] \in \mathrm{G}(n-1, V) \mid V_{n-3} \subset \Pi \subset V_n\} \cong \mathbb{P}^2\ (\sigma\text{-plane})
\end{aligned}
\tag{4.1}
$$

with some $V_{n-2} \subset V$ and $V_{n-3} \subset V_n \subset V$, respectively. As displayed above, we call these planes *ρ-plane* and *σ-plane*, respectively. It is easy to deduce the following proposition:

Proposition 4.1. *In $\mathrm{G}(3, \wedge^{n-1}V)$, the set of ρ-planes $\overline{\mathscr{P}}_\rho$ and the set of σ-planes $\overline{\mathscr{P}}_\sigma$ are given by*

$$
\begin{aligned}
\overline{\mathscr{P}}_\rho &= \left\{ \left[(V/V_{n-2}) \wedge \left(\wedge^{n-2} V_{n-2}\right) \right] \mid [V_{n-2}] \in \mathrm{G}(n-2, V) \right\}\\
\overline{\mathscr{P}}_\sigma &= \left\{ \left[\wedge^2 (V_n/V_{n-3}) \wedge (\wedge^{n-3} V_{n-3}) \right] \mid [V_{n-3} \subset V_n] \in \mathrm{F}(n-3, n, V) \right\},
\end{aligned}
$$

where $\overline{\mathscr{P}}_\rho \simeq \mathrm{G}(n-2, V)$ and $\overline{\mathscr{P}}_\sigma \simeq \mathrm{F}(n-3, n, V)$.

Let us make the following definition:

Definition 4.2. We call a conic q in $\mathrm{G}(n-1, V)$ a *τ-conic* if $\mathbb{P}_q \cap \mathrm{G}(n-1, V) = q$. A conic q is called a *ρ-conic* and *σ-conic* if the plane $\mathbb{P}_q$ is contained in $\mathrm{G}(n-1, V)$, and in that case $\mathbb{P}_q$ is called a ρ-plane and σ-plane, respectively.

Let us denote by $[Q_y]$ the image of $y \in \mathscr{Y}$ under $\mathscr{Y} \to \mathscr{H}$. By slight abuse of terminology, we say y is a rank k point if $\operatorname{rank} Q_y = k$. By Proposition 3.7, the fiber of $\mathscr{X} \to \mathscr{Y}$ over a rank 3 or 4 point y is a conic, which we denote it by q_y.

Proposition 4.3. (1) *If* $\operatorname{rank} Q_y = 4$*, then q_y is a τ-conic.* (2) *If* $\operatorname{rank} Q_y = 3$*, then the plane $\mathbb{P}_{q_y}$ is a ρ-plane, hence q_y is a ρ-conic.*

Proof. (1) If q_y is a ρ-conic, then $(n-2)$-planes in Q_y parameterized by q_y must contain a $\mathbb{P}(V_{n-2})$ in common but this can not be the case. If q_y is a σ-conic, then $(n-2)$-planes in Q_y parameterized by q_y must be contained in one $\mathbb{P}(V_n)$ but this also can not be the case. Hence q_y is a τ-conic. The claim (2) is clear since the planes parametrized by q_y contain the vertex $\mathbb{P}(V_{n-2})$ of Q_y in common. □

Example 4.4. (Smooth Conics) Taking a basis $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$ of V, consider the subspaces $V_{n-3} = \langle \mathbf{e}_4, \dots, \mathbf{e}_n \rangle$, $V_n = \langle \mathbf{e}_1, \dots, \mathbf{e}_n \rangle$ and $V_{n-2} = \langle \mathbf{e}_4, \dots, \mathbf{e}_{n+1} \rangle$. An example of τ-conic may be given

$$q_\tau = \left\{ [s\mathbf{e}_1 + t\mathbf{e}_2, s\mathbf{e}_3 + t\mathbf{e}_4, \mathbf{e}_5, \dots, \mathbf{e}_{n+1}] \mid [s,t] \in \mathbb{P}^1 \right\}.$$

Similarly, as a $\mathbb{P}^1$-family of planes in the ρ-plane $\mathrm{P}_{V_{n-2}}$ and τ-plane $\mathrm{P}_{V_{n-3}V_n}$, respectively, we have the following examples:

$$q_\rho = \left\{ [s^2\mathbf{e}_1 + st\mathbf{e}_2 + t^2\mathbf{e}_3, \mathbf{e}_4, \dots, \mathbf{e}_{n+1}] \right\}, \ q_\sigma = \left\{ [s\mathbf{e}_1 + t\mathbf{e}_2, s\mathbf{e}_2 + t\mathbf{e}_3, \mathbf{e}_4, \dots, \mathbf{e}_n] \right\},$$

where $[s,t] \in \mathbb{P}^1$ parameterizes each conic q. $\square$

Example 4.5. (Rank two conics) Since a line in $\mathrm{G}(n-1, V)$ takes the form $l_{V_{n-2}V_n} = \{[\Pi] \mid V_{n-2} \subset \Pi \subset V_n\}$ with some $V_{n-2} \subset V_n \subset V$, reducible conics q have the following form:

$$q = l_{V_{n-2}V_n} \cup l_{V'_{n-2}V'_n} \tag{4.2}$$

with

- $\dim(V_{n-2} \cap V'_{n-2}) \geq n-3$,
- $V_{n-2}, V'_{n-2} \subset V_n \cap V'_n$, and
- $V_{n-2} \neq V'_{n-2}$ or $V_n \neq V'_n$.

These conics will be described in detail in the section 5.

Descriptions of rank one conics may be found in Appendix A.

4.2. **Hilbert scheme $\mathscr{Y}_0$ of conics on** $\mathrm{G}(n-1, V)$**.** Consider a point $[U] \in \mathrm{G}(3, \wedge^{n-1} V)$. To describe conics in $\mathrm{G}(n-1, V) \subset \mathbb{P}(3, \wedge^{n-1} V)$, it suffices to find a condition for a plane $\mathbb{P}(U)$ to be contained in $\mathrm{G}(n-1, V)$ or cut out a conic from $\mathrm{G}(n-1, V)$. For this, we introduce the composite φ of the following maps:

$$\varphi \colon \mathsf{S}^2(\wedge^{n-1} V) \simeq \mathsf{S}^2(\wedge^2 V^*) \xrightarrow{\psi} \wedge^4 V^*, \tag{4.3}$$

where the first map is induced by the duality $\wedge^{n-1} V \simeq \wedge^2 V^*$ coming from the wedge product pairing $\wedge^{n-1} V \times \wedge^2 V \to \wedge^{n+1} V$, and ψ is induced by the wedge product. Note that the zero locus of ψ is nothing but $\mathrm{G}(2, V^*)$ since we obtain the Plücker quadrics defining $\mathrm{G}(2, V^*)$ by writing ψ with coordinates. Moreover, the duality $\wedge^{n-1} V \simeq \wedge^2 V^*$ induces an isomorphism $\mathrm{G}(n-1, V) \simeq \mathrm{G}(2, V^*)$. Therefore $\mathrm{G}(n-1, V)$ is the zero locus of φ.

Now we consider the restriction of φ to a 3-plane $U \subset \wedge^{n-1} V$:

$$\varphi_U := \varphi|_{\mathsf{S}^2 U} \colon \mathsf{S}^2 U \to \wedge^4 V^*.$$

Let U' be the 3-plane of $\wedge^2 V^*$ corresponding to U and denote by $\psi_{U'}$ the restriction of ψ to U'. Since $\mathrm{G}(2, V^*)$ is the zero locus of ψ, $\mathbb{P}(U') \subset \mathrm{G}(2, V^*)$ iff $\psi_{U'} = 0$. Similarly, $\mathbb{P}(U') \cap \mathrm{G}(2, V^*)$ is a conic iff the restrictions of the Plücker quadrics on $\mathbb{P}(\mathsf{S}^2 U'^*)$ form a point, i.e., one-dimensional subspace of $\mathsf{S}^2 U'^*$, which is equivalent to the condition $\operatorname{rank} \psi_{U'} = 1$. Translating this, we immediately obtain the following descriptions on the intersection $\mathbb{P}(U) \cap \mathrm{G}(n-1, V)$:

Proposition 4.6. *For a* 3*-plane* $U \subset \wedge^{n-1} V$*,* $\mathbb{P}(U) \cap \mathrm{G}(n-1, V)$ *contains a conic iff* $\operatorname{rank} \phi_U \leq 1$*. Moreover, the following properties hold:*

(1) $\left\{ [U] \in \mathrm{G}(3, \wedge^{n-1} V) \mid \varphi_U = 0 \right\} = \overline{\mathscr{P}}_\rho \sqcup \overline{\mathscr{P}}_\sigma$.

(2) *If* $\operatorname{rank} \varphi_U = 1$*, then* $\mathbb{P}(U) \cap \mathrm{G}(n-1, V)$ *is a conic which is the zero locus of* φ_U*.*

Motivated from the above descriptions of conics, we define the following scheme with reduced structure:

$$\mathscr{Y}_0 := \left\{([U],[c_U]) \mid [U] \in \mathrm{G}(n-1,V), [c_U] \in \mathbb{P}(\mathsf{S}^2U^*) \text{ s.t. } (c_U)_0 \subset (\varphi_U)_0\right\}, \tag{4.4}$$

where $(c_U)_0$ and $(\varphi_U)_0$ represents the zero locus in $\mathbb{P}(U)$ of c_U and φ_U, respectively.

Theorem 4.7. *$\mathscr{Y}_0$ is smooth and isomorphic to the Hilbert scheme of conics on $\mathrm{G}(n-1,V)$.*

Proof. By definition, $\mathscr{Y}_0$ obviously parameterizes conics in $\mathrm{G}(n-1,V)$ in one to one way. Moreover, there is a family in $\mathbb{P}(\wedge^{n-1}V) \times \mathscr{Y}_0$ of corresponding conics $(c_U)_0$ at each point $([U],[c_U]) \in \mathscr{Y}_0$. Therefore, by the universal property of the Hilbert scheme, there is a unique map from $\mathscr{Y}_0$ to the Hilbert scheme $\mathrm{Hilb}^{\mathrm{co}}\mathrm{G}(n-1,V)$ of conics in $\mathrm{G}(n-1,V)$. Since the smoothness of the Hilbert scheme is known in [16] and [3], we have $\mathscr{Y}_0 \simeq \mathrm{Hilb}^{\mathrm{co}}\mathrm{G}(n-1,V)$. $\square$

Let us consider the natural projection $\mathscr{Y}_0 \to \mathrm{G}(n-1,V)$ and denote by $\overline{\mathscr{Y}}$ its image with the reduced structure. Let $\nu\colon \overline{\mathscr{Y}}' \to \overline{\mathscr{Y}}$ be the normalization (one should be able to show that $\overline{\mathscr{Y}}$ is normal in general extending the explicit description given in [12] for $n=4$). The following descriptions of $\overline{\mathscr{Y}}$ and related properties are easy to derive:

Proposition 4.8. (1) *We have*

$$\overline{\mathscr{Y}} = \left\{[U] \in \mathrm{G}(3,\wedge^{n-1}V) \mid \operatorname{rank}\varphi_U \le 1\right\}.$$

(2) *$\mathscr{Y}_0 \to \overline{\mathscr{Y}}'$ is isomorphism outside $\nu^{-1}\overline{\mathscr{P}}_\rho$ and $\nu^{-1}\overline{\mathscr{P}}_\sigma$.*
(3) *Let G_ρ and F_σ be the exceptional set over $\nu^{-1}\overline{\mathscr{P}}_\rho$ and $\nu^{-1}\overline{\mathscr{P}}_\sigma$, respectively. Then $G_\rho \to \nu^{-1}\overline{\mathscr{P}}_\rho$ and $F_\sigma \to \nu^{-1}\overline{\mathscr{P}}_\sigma$ are $\mathbb{P}^5$-bundles whose fiber parameterizes ρ- or σ-conics in a fixed ρ- or σ-plane respectively.*

4.3. **Small resolution $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$.** We find a small resolution $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$ by translating the condition $\operatorname{rank}\varphi_U \le 1$ into an equivalent form. For each $v \in V$, let us define a linear map $E_v : \wedge^{n-1}V \to \wedge^n V$ by $u \mapsto v \wedge u$. Consider the restriction $E_v|_U$ to $U \subset \wedge^{n-1}V$ and introduce

$$a_U = \{v \in V \mid E_v|_U = 0\},$$

which is nothing but the annihilator of U. Note that $\dim U = 3$ implies $\dim a_U \le n-2$. We prove the following proposition in Appendix A.

Proposition 4.9. *For $[U] \in \mathrm{G}(3,\wedge^{n-1}V)$, $\dim a_U \ge n-3 \Longleftrightarrow \operatorname{rank}\varphi_U \le 1$.*

By this proposition, it is immediate to see that

$$\overline{\mathscr{Y}} = \left\{[U] \in \mathrm{G}(3,\wedge^{n-1}V) \mid \dim a_U \ge n-3\right\}.$$

Below we define a Springer type resolution $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$, which turns out to be a small resolution.

Definition 4.10. For $n \ge 3$, we define

$$\mathscr{Y}_3 = \{([U],[V_{n-3}]) \mid V_{n-3} \subset a_U\} \subset \mathrm{G}(3,\wedge^{n-1}V) \times \mathrm{G}(n-3,V),$$

where $\mathrm{G}(n-3,V)$ should be understood as one point when $n=3$. Obviously, the image of the projection of $\mathscr{Y}_3$ to the first factor coincides with $\overline{\mathscr{Y}}$.

Since $E_v|_U = 0$ ($\forall v \in V_{n-3}$) implies that U is the $\mathbb{C}$-span of non-vanishing vectors of the form $\bar{u}_i \wedge v_1 \wedge \cdots \wedge v_{n-3}$ ($i = 1, 2, 3$) with $\bar{u}_i \in \wedge^2(V/V_{n-3})$ and $v_1, \ldots, v_{n-3}$ being a basis of V_{n-3}, the fiber of the natural projection $\mathscr{Y}_3 \to \mathrm{G}(n-3, V)$ over $[V_{n-3}] \in \mathrm{G}(n-3, V)$ can be identified with $\mathrm{G}(3, \wedge^2(V/V_{n-3}))$. Hence we see that

$$\mathscr{Y}_3 = \mathrm{G}(3, \wedge^2 \mathfrak{Q}),$$

and in particular $\mathscr{Y}_3$ is smooth.

Proposition 4.11. *The morphism $\rho_{\mathscr{Y}_3} : \mathscr{Y}_3 \to \overline{\mathscr{Y}}'$ is isomorphic over $\overline{\mathscr{Y}}' \setminus \nu^{-1}\overline{\mathscr{P}}_\rho$ and is a small resolution with $\rho_{\mathscr{Y}_3}^{-1}(x) \simeq \mathbb{P}^{n-3}$ for each $x \in \nu^{-1}\overline{\mathscr{P}}_\rho$. In particular, $\rho_{\mathscr{Y}_3}$ is an isomorphism if $n = 3$, and $\nu^{-1}\overline{\mathscr{P}}_\rho = \mathrm{Sing}\,\overline{\mathscr{Y}}'$ if $n \geq 4$.*

Proof. It is easy to see that the fiber of $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$ over each point of $\nu^{-1}\overline{\mathscr{P}}_\rho$ is $\mathrm{G}(n-3, n-2) \simeq \mathbb{P}^{n-3}$, and $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$ is bijective over $\overline{\mathscr{Y}}' \setminus \nu^{-1}\overline{\mathscr{P}}_\rho$. □

Remark 4.12. In case $n = 3$, we have $\mathscr{Y}_3 = \overline{\mathscr{Y}}' = \overline{\mathscr{Y}} = \mathrm{G}(3, \wedge^2 V)$.

4.4. **Small resolution $\widetilde{\mathscr{Y}} \to \overline{\mathscr{Y}}'$ via the Hilbert scheme $\mathscr{Y}_0$.** We construct another small resolution $p_{\widetilde{\mathscr{Y}}} : \widetilde{\mathscr{Y}} \to \overline{\mathscr{Y}}'$ for $n \geq 4$, which is the (anti-)flip of $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$. We give $\widetilde{\mathscr{Y}}$ from $\mathscr{Y}_0$ by contracting the exceptional set (divisor) over $\nu^{-1}\overline{\mathscr{P}}_\sigma$.

Let R_ρ (resp. R_σ) be the extremal ray spanned by lines in fibers of $G_\rho \to \nu^{-1}\overline{\mathscr{P}}_\rho$ (resp. $F_\sigma \to \nu^{-1}\overline{\mathscr{P}}_\sigma$). We show that $R_\rho \neq R_\sigma$. Indeed, note that F_σ is a prime divisor and $G_\rho \cap F_\sigma = \emptyset$. Therefore, $F_\sigma \cdot R_\rho = 0$ and $F_\sigma \cdot R_\sigma < 0$ and hence $R_\rho \neq R_\sigma$. Since $\overline{\mathscr{Y}}'$ is smooth along $\overline{\mathscr{P}}_\sigma$ by Proposition 4.11, the discrepancy of F_σ is positive and then R_σ is $K_{\mathscr{Y}_0}$-negative. Therefore there exists a unique extremal contraction $\mathscr{Y}_0 \to \widetilde{\mathscr{Y}}$ over $\overline{\mathscr{Y}}'$ associated to R_σ, which is nothing but the contraction of F_σ. We denote by G_σ the image of F_σ.

The following proposition follows from the above construction of $\widetilde{\mathscr{Y}}$:

Proposition 4.13. *$\widetilde{\mathscr{Y}}$ parameteirizes τ- and ρ-conics, and σ-planes.*

We retain the notation G_ρ to represent the locus in $\widetilde{\mathscr{Y}}$ parameterizing ρ-conics and denote by $\mathcal{Q}_\rho$ the universal quotient bundle on $\mathrm{G}(n-2, V)$.

Proposition 4.14. *G_ρ is isomorphic to $\mathbb{P}(\mathsf{S}^2\mathcal{Q}_\rho^*)$. It is also isomorphic to $\widetilde{\mathrm{S}}_3$.*

Proof. The first claim is clear since $\mathbb{P}(\mathcal{Q}_\rho) \to \overline{\mathscr{P}}_\rho \simeq \mathrm{G}(n-2, V)$ is the family of ρ-planes. The second one follows from the definition of the resolution $p_{\widetilde{\mathrm{S}}_3} : \widetilde{\mathrm{S}}_3 \to \mathrm{S}_3$ (see Proposition 2.1). □

Proposition 4.15. *$p_{\widetilde{\mathscr{Y}}} : \widetilde{\mathscr{Y}} \to \overline{\mathscr{Y}}'$ is a small resolution for $n \geq 4$, and is the blow-up along $\nu^{-1}\overline{\mathscr{P}}_\rho$ for $n = 3$. Non-trivial fibers of $p_{\widetilde{\mathscr{Y}}}$ are copies of $\mathbb{P}^5$.*

Proof. $\widetilde{\mathscr{Y}}$ is smooth since $\mathscr{Y}_0$ is smooth by Theorem 4.7 and $\overline{\mathscr{Y}}'$ is smooth along $\nu^{-1}\overline{\mathscr{P}}_\sigma$ by Proposition 4.11.

Note that G_ρ is the $p_{\widetilde{\mathscr{Y}}}$-exceptional locus since the restriction of $p_{\widetilde{\mathscr{Y}}}|_{G_\rho}$ is a $\mathbb{P}^5$-bundle over $\nu^{-1}\overline{\mathscr{P}}_\rho \simeq \overline{\mathscr{P}}_\rho$. If $n \geq 4$, then G_ρ is not a divisor by dimension count. In case $n = 3$, G_ρ is a prime divisor. Since $\overline{\mathscr{Y}}'$ is smooth by Proposition 4.11, and $G_\rho \to \nu^{-1}\overline{\mathscr{P}}_\sigma$ is a $\mathbb{P}^5$-bundle, we see that $K_{\widetilde{\mathscr{Y}}} = p_{\widetilde{\mathscr{Y}}}^* K_{\overline{\mathscr{Y}}'} + 5G_\rho$. Let $p'_{\widetilde{\mathscr{Y}}} : \widetilde{\mathscr{Y}}' \to \overline{\mathscr{Y}}'$ be the blow-up along $\nu^{-1}\overline{\mathscr{P}}_\rho$ and G'_ρ the $p'_{\widetilde{\mathscr{Y}}}$-exceptional divisor. Then we have

$K_{\widetilde{\mathscr{Y}}'} = p'^{*}_{\widetilde{\mathscr{Y}}} K_{\overline{\mathscr{Y}}'} + 5G'_\rho$. It is well-known that there is only one valuation of $k(\overline{\mathscr{Y}}')$ associated to exceptional divisors with center $\nu^{-1}\overline{\mathscr{P}}_\rho$ and discrepancy 5. Therefore, $\widetilde{\mathscr{Y}}$ and $\widetilde{\mathscr{Y}}'$ are isomorphic in codimension one. Moreover, since $-K_{\widetilde{\mathscr{Y}}}$ and $-K_{\widetilde{\mathscr{Y}}'}$ are relatively ample over $\overline{\mathscr{Y}}'$, $\widetilde{\mathscr{Y}}$ and $\widetilde{\mathscr{Y}}'$ must be isomorphic by [25, Lemma 5.5]. □

4.5. **Rational map $\mathscr{Y}_3 \dashrightarrow \mathscr{H}$ via double spin decomposition.** Consider a point $([U],[V_{n-3}]) \in \mathscr{Y}_3 = \mathrm{G}(3, \wedge^2\mathfrak{Q})$ with $[U] \in \mathrm{G}(3, \wedge^2(V/V_{n-3}))$. To describe $\wedge^3 U$, we use the following irreducible decomposition as $sl(V/V_{n-3})$-modules (see [6, §19.1] for example):

$$\begin{aligned}\wedge^3\,(\wedge^2(V/V_{n-3})) = \\ \mathsf{S}^2(V/V_{n-3}) \otimes \det(V/V_{n-3}) \oplus \mathsf{S}^2(V/V_{n-3})^* \otimes \det(V/V_{n-3})^{\otimes 2}.\end{aligned} \tag{4.5}$$

We will call this "double spin" decomposition since the symmetric powers in the r.h.s. are identified with $V_{2\lambda_s}$ and $V_{2\lambda_{\bar{s}}}$ as $so(\wedge^2 V/V_{n-3})(\simeq sl(V/V_{n-3}))$-modules, where λ_s and $\lambda_{\bar{s}}$ represent the spinor and conjugate spinor weights, respectively (see [*loc. cit.*]). Considering this decomposition fiberwise in the projective bundle $\mathbb{P}(\wedge^3(\wedge^2\mathfrak{Q}))$ over $\mathrm{G}(n-3,V)$, we have the following sequence of (rational) maps:

$$\begin{aligned}\mathscr{Y}_3 \hookrightarrow & \mathbb{P}(\mathsf{S}^2\mathfrak{Q} \otimes \mathcal{O}_{\mathrm{G}(n-3,V)}(-1) \oplus \mathsf{S}^2\mathfrak{Q}^*) \\ & \dashrightarrow \mathscr{U} = \mathbb{P}(\mathsf{S}^2\mathfrak{Q}^*) \hookrightarrow \mathbb{P}(\mathsf{S}^2 V^* \otimes \mathcal{O}_{\mathrm{G}(n-3,V)}),\end{aligned} \tag{4.6}$$

where the rational map in the middle is the projection to the second factor and the last inclusion comes from the surjection $V \otimes \mathcal{O}_{\mathrm{G}(n-3,V)} \to \mathfrak{Q} \to 0$. We further consider the natural projection $\mathbb{P}(\mathsf{S}^2 V^* \otimes \mathcal{O}_{\mathrm{G}(n-3,V)}) \to \mathbb{P}(\mathsf{S}^2 V^*)$. Then the image of the composite is contained in the locus $\mathscr{H}$ of the quadrics of rank ≤ 4, and hence we have a rational map

$$\phi \colon \mathscr{Y}_3 \dashrightarrow \mathscr{H}\ (:= \mathrm{S}_4).$$

To obtain a morphism, we consider the inverse images $\mathscr{P}_\rho, \mathscr{P}_\sigma$ of $\nu^{-1}\overline{\mathscr{P}}_\rho$ and $\nu^{-1}\overline{\mathscr{P}}_\sigma$, respectively, under the resolution $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$. Then it is clear from the definitions that

$$\mathscr{P}_\rho \simeq \mathrm{F}(n-3, n-2; V) \simeq \mathbb{P}(\mathfrak{Q}), \quad \mathscr{P}_\sigma \simeq \mathrm{F}(n-3, n; V) \simeq \mathbb{P}(\mathfrak{Q}^*). \tag{4.7}$$

Proposition 4.16. *Under the embedding $\mathscr{Y}_3 \subset \mathbb{P}(\mathsf{S}^2\mathfrak{Q} \otimes \mathcal{O}_{\mathrm{G}(n-3,V)}(-1) \oplus \mathsf{S}^2\mathfrak{Q}^*)$, $\mathscr{P}_\rho$ and $\mathscr{P}_\sigma$ are identified with*

$$\mathscr{P}_\rho = v_2(\mathbb{P}(\mathfrak{Q})), \quad \mathscr{P}_\sigma = v_2(\mathbb{P}(\mathfrak{Q}^*)).$$

Moreover, $\mathscr{P}_\rho = \mathscr{Y}_3 \cap \mathbb{P}(\mathsf{S}^2\mathfrak{Q} \otimes \mathcal{O}_{\mathrm{G}(n-3,V)}(-1))$ scheme-theoretically.

Proof. The claims follows from the decomposition (4.5) and its explicit description given in Proposition B.1, (B.3). □

Definition 4.17. We define $\rho_{\mathscr{Y}_2} \colon \mathscr{Y}_2 \to \mathscr{Y}_3$ to be the blow-up along $\mathscr{P}_\rho$, and denote by F_ρ its exceptional divisor.

Clearly there is a morphism $\mathscr{Y}_2 \to \mathrm{G}(n-3,V)$ as well as $\mathscr{Y}_3 \to \mathrm{G}(n-3,V)$.

4.6. **The case $n = 3$ ($\dim V = 4$).** When $n = 3$, projective bundles over $\mathrm{G}(n-3, V)$ reduce to the corresponding projective spaces, and considerable simplifications may be observed, for example, in

$$\mathscr{Y}_3 = \overline{\mathscr{Y}}' = \overline{\mathscr{Y}} = \mathrm{G}(3, \wedge^2 V) \text{ and } \mathscr{P}_\rho = v_2(\mathbb{P}(V)) \subset \mathbb{P}(\mathsf{S}^2 V).$$

Also in this case, we have $\mathscr{Y}_2 \simeq \widetilde{\mathscr{Y}}$ by Propositions 4.11 and 4.15. Then the birational morphism $\phi\colon \mathscr{Y}_3 \dashrightarrow \mathscr{H} (= \mathbb{P}(\mathsf{S}^2 V^*))$ lifts to a morphism $\widetilde{\phi}\colon \mathscr{Y}_2 \to \mathscr{H}$ by the last assertion in Proposition 4.16.

In this subsection, we study the case $n = 3\,(\dim V = 4)$ (where $\mathscr{H} = \mathbb{P}(\mathsf{S}^2 V^*)$). The results below will be used to study the case of $n \geq 4\,(\dim V \geq 5)$ (where $\mathscr{H} = \mathrm{S}_4 \subset \mathbb{P}(\mathsf{S}^2 V^*)$) in the next subsection. Also these will be used extensively in [14].

Proposition 4.18. (1) *The Stein factorization of $\widetilde{\phi}\colon \widetilde{\mathscr{Y}} \to \mathscr{H}$ factors through the double cover $\rho_{\mathscr{Y}}: \mathscr{Y} \to \mathscr{H}$.*

(2) *Let $\rho_{\widetilde{\mathscr{Y}}}: \widetilde{\mathscr{Y}} \to \mathscr{Y}$ be the induced morphism. Then $\rho_{\widetilde{\mathscr{Y}}}$ is birational and a $K_{\widetilde{\mathscr{Y}}}$-negative extremal divisorial contraction.*

(3) *Let $F_{\widetilde{\mathscr{Y}}}$ be the $\rho_{\widetilde{\mathscr{Y}}}$-exceptional divisor. Then the image of $F_{\widetilde{\mathscr{Y}}}$ by $\rho_{\widetilde{\mathscr{Y}}}$ coincides with $G_{\mathscr{Y}}$, and $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-fibration outside $G^1_{\mathscr{Y}}$.*

(4) *It holds that*

$$K_{\widetilde{\mathscr{Y}}} = \rho^*_{\widetilde{\mathscr{Y}}} K_{\mathscr{Y}} + F_{\widetilde{\mathscr{Y}}}.$$

In particular, $\mathscr{Y}$ has only terminal singularities with $\operatorname{Sing} \mathscr{Y} = G_{\mathscr{Y}}$.

(5) *Let $w = (w_{kl})$ be the 4×4 matrix representing $[Q] \in \mathbb{P}(\mathsf{S}^2 V^*)$. Then the fiber of $\tilde{\phi}$ is described according to the rank of w as follow:*
   (a) *When $\operatorname{rank} w = 4$, $\widetilde{\phi}^{-1}([Q])$ consists of two points.*
   (b) *When $\operatorname{rank} w = 3$, $\widetilde{\phi}^{-1}([Q])$ consists of one point.*
   (c) *When $\operatorname{rank} w = 2$, $\widetilde{\phi}^{-1}([Q]) \simeq \mathbb{P}^1 \times \mathbb{P}^1$.*
   (d) *When $\operatorname{rank} w = 1$, $\tilde{\phi}^{-1}([Q]) \simeq \mathbb{P}(1^3, 2)$. The vertex of $\tilde{\phi}^{-1}([Q])$ corresponds to the σ-plane P_{V_3}, where $Q = 2\mathbb{P}(V_3)$, and $\tilde{\phi}^{-1}([Q]) \cap F_\rho \simeq \mathbb{P}^2$ which is a hyperplane section of $\mathbb{P}(1^3, 2) \subset \mathbb{P}^6$.*

Proof. Let $\widetilde{\mathscr{Y}} \to \mathscr{Y}' \to \mathscr{H}$ be the Stein factorization of $\widetilde{\phi}$. We denote by $\rho_{\widetilde{\mathscr{Y}}}$ and $F_{\widetilde{\mathscr{Y}}}$, the induced morphism $\widetilde{\mathscr{Y}} \to \mathscr{Y}'$ and the $\rho_{\widetilde{\mathscr{Y}}}$-exceptional locus respectively (this notation will be compatible with (2) and (3) after showing that the induced morphism $\mathscr{Y}' \to \mathscr{H}$ coincides with the double cover $\rho_{\mathscr{Y}}: \mathscr{Y} \to \mathscr{H}$).

Let us start with showing that $\widetilde{\phi}(F_\rho) = \mathrm{S}_3$. Let Q be a rank three quadric Q in $\mathbb{P}(V)$. Then, from (I.3) in Appendix B, $[Q]$ cannot be in the image of ϕ. Therefore the locus S_3 is contained in $\widetilde{\phi}(F_\rho)$. Since F_ρ and S_3 are prime divisors in $\widetilde{\mathscr{Y}}$ and $\mathscr{H}$ respectively, it holds that $\widetilde{\phi}(F_\rho) = \mathrm{S}_3$.

Proof of (5) (a). Let Q be a rank four quadric Q in $\mathbb{P}(V)$, i.e., $[Q] \in \mathrm{S}_4 \setminus \mathrm{S}_3$. From (I.2) in Appendix B, $\phi^{-1}([Q])$ consists of two points $[v, w]$ satisfying $v.w = \pm\sqrt{\det w}\,\mathrm{id}_4$. Since $\widetilde{\phi}(F_\rho) = \mathrm{S}_3$, $\widetilde{\phi}^{-1}([Q])$ also consists of two points.

We know now that $\mathscr{Y}' \to \mathscr{H}$ is a finite morphism of degree two, and its branch locus is contained in S_3.

Proof of a weaker assertion than (5) (c). Let Q be a rank two quadric Q in $\mathbb{P}(V)$ and w an associated symmetric matrix. We show that $\tilde{\phi}^{-1}([Q])$ contains a $\mathbb{P}^1 \times \mathbb{P}^1$. Changing the coordinate of V suitably, we may assume that $[w]$ is

given in the form $w_0 = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & O_2 \\ O_2 & O_2 \end{pmatrix}$ with O_2 being the 2×2 zero matrix. Then by the properties (I.4) and (I.2), we obtain $v = \begin{pmatrix} O_2 & O_2 \\ O_2 & \begin{smallmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{smallmatrix} \end{pmatrix}$. Now substituting $[v, w] = [v, tw_0]$ $(t \neq 0)$ into the equation in the first line of (B.3), we have

$$v_{11}v_{22} - v_{12}^2 + t^2 = 0 \ \ (t \neq 0).$$

The closure S of this locus in $\mathscr{Y}_3 = \mathrm{G}(3, \wedge^2 V)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that the restriction of the blow-up $\widetilde{\mathscr{Y}} \to \mathscr{Y}_3$ over $S \subset \mathscr{Y}_3$ is the blow-up along the locus $t = 0$. Hence the strict transform S' of S in $\widetilde{\mathscr{Y}}$ is also isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that S' is contained in the fiber of the restriction over S.

Proof of a similar statement to (2) for $\widetilde{\mathscr{Y}} \to \mathscr{Y}'$. Since $\rho(\widetilde{\mathscr{Y}}) = 2$, we have $\rho(\widetilde{\mathscr{Y}}/\mathscr{Y}') \leq 1$. Moreover, since the fiber over a rank two point is at least 2-dimensional and $\dim \mathrm{S}_2 = 6$, $F_{\widetilde{\mathscr{Y}}}$ is a prime divisor. We see that the contraction $\rho_{\widetilde{\mathscr{Y}}}$ is $K_{\widetilde{\mathscr{Y}}}$-negative by computing the intersection number between $K_{\widetilde{\mathscr{Y}}}$ and a ruling of S'. Thus $\mathscr{Y}'$ has only terminal singularities.

Proof of (1). $\mathscr{Y}'$ is Cohen-Macaulay since it is terminal and hence $\mathscr{Y}' \to \mathscr{H}$ is flat. Then its branch locus is empty or a divisor but the former case cannot occur since $\mathscr{H} = \mathbb{P}(\mathsf{S}^2 V^*)$ is simply connected. Therefore the branch locus of $\mathscr{Y}' \to \mathscr{H}$ coincides with S_3. Now we see that $\mathscr{Y}' \simeq \mathscr{Y}$ since both $\mathscr{Y}' \to \mathscr{H}$ and $\mathscr{Y} \to \mathscr{H}$ are both flat, finite of degree two and are branched along S_3.

Proof of (5) (b). Since S_3 is the branch locus, $F_\rho \to \mathrm{S}_3$ is birational. Therefore we see that the fiber over a rank three points consists of one point as claimed.

We have shown (1), (2), the first half of (3) and (5) (b). The second half of (3) will follow from (5) (c).

We will show two resolutions $F_\rho \to \mathrm{S}_3$ and $\widetilde{\mathrm{S}}_3 \to \mathrm{S}_3$ coincides with each other. First we note that $\rho(F_\rho) = \rho(\widetilde{\mathrm{S}}_3) = 2$ and then $\rho(F_\rho/\mathrm{S}_3) = \rho(\widetilde{\mathrm{S}}_3/\mathrm{S}_3) = 1$. Since S_3 is $\mathbb{Q}$-factorial, $F_\rho \to \mathrm{S}_3$ is a divisorial contraction. Let G_1 and G_2 be the exceptional divisors of $F_\rho \to \mathrm{S}_3$ and $\widetilde{\mathrm{S}}_3 \to \mathrm{S}_3$ respectively. Since $\widetilde{\mathrm{S}}_3 \to \mathrm{S}_3$ is a crepant resolution, the valuation of G_2 in $k(\mathrm{S}_3)$ is a unique crepant valuation. If the discrepancy of G_1 is positive, then we see that any exceptional valuation in $k(\mathrm{S}_3)$ must have positive discrepancy by computing the discrepancies of exceptional divisors over F_ρ, which is a contradiction to the existence of G_2. Therefore $F_\rho \to \mathrm{S}_3$ is crepant, and moreover the valuations of G_1 and G_2 are the same by the uniqueness of the crepant valuation. In particular, $F_\rho \to \mathrm{S}_3$ and $\widetilde{\mathrm{S}}_3 \to \mathrm{S}_3$ are isomorphic in codimension one. Note that $-G_1$ and $-G_2$ are relatively ample over S_3. Let $p\colon \Gamma \to F_\rho$ and $q\colon \Gamma \to \mathrm{S}_3$ be a common resolution of F_ρ and S_3. Thus, by the standard argument using the negativity lemma, we see that $p^*(-G_1) = q^*(-G_2)$. This implies that two resolutions $F_\rho \to \mathrm{S}_3$ and $\widetilde{\mathrm{S}}_3 \to \mathrm{S}_3$ coincides with each other.

Proof of (5) (c). As we see above, the fiber over a rank two point $[Q]$ contains at least $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The fiber of $F_\rho \to \mathrm{S}_3$ over $[Q]$ is isomorphic to $\mathbb{P}^1$ by the description of the fibers of $\widetilde{\mathrm{S}}_3 \to \mathrm{S}_3$. Thus the fiber $\tilde{\phi}^{-1}([Q])$ coincides with S'.

Proof of (4). We obtain the claimed formula by computing the intersection number between $K_{\widetilde{\mathscr{Y}}}$ and a ruling of S'.

Proof of (5) (d). Let Q be a rank one quadric in $\mathbb{P}(V)$ and w an associated symmetric matrix. Then w can be written as $(a_k a_l)$ with some $a \in V^*$. Then from (I.5) in Appendix, we see that $\operatorname{rank} v \leq 1$. Writing $v_{ij} = x_i x_j$ with $x \in V$ and also

solving (B.3) we obtain

$$\phi^{-1}([Q]) = \{[x_i x_j, t a_k a_l] \mid a.x = 0, t \neq 0\}\,. \tag{4.8}$$

The closure of this locus in $\mathscr{Y}_3$ is isomorphic to the cone over $v_2(\mathbb{P}^2) \simeq \mathbb{P}^2$ from the vertex $[0, a_k a_l] \in \mathbb{P}(\mathsf{S}^2 V \oplus \mathsf{S}^2 V^*)$, which is isomorphic to $\mathbb{P}(1^3, 2)$. Then we have the former assertion (5) (d) by a similar argument in case $\operatorname{rank} w = 2$. The latter assertion is clear from the above description.

□

Remark 4.19. It is convenient to give a coordinate-free description of $\widetilde{\phi}^{-1}([Q])$ in case $\operatorname{rank} Q = 1$. Instead of $\widetilde{\phi}^{-1}([Q])$, we may describe its isomorphic image $\Phi \subset \mathscr{Y}_3$ under $\widetilde{\mathscr{Y}} = \mathscr{Y}_2 \to \mathscr{Y}_3$. Note that Φ is the closure in $\mathscr{Y}_3$ of $\phi^{-1}([Q])$ and its equation is given by (4.8). Let $Q = 2\mathbb{P}(V_3)$ as in Proposition 4.18 (5) (d). The vertex of $\widetilde{\phi}^{-1}([Q])$ corresponds to the σ-plane $\mathrm{P}_{V_3} = \{\mathbb{C}^2 \subset V_3\}$. By the equation (4.8), points $[\mathrm{P}_{V_1}]$ which correspond to ρ-planes and are contained in Φ satisfy $V_1 \subset V_3$. Since Φ is the cone over the Veronese surface $v_2(\mathbb{P}(V_3))$, it is swept out by lines joining $[\mathrm{P}_{V_3}]$ and $[\mathrm{P}_{V_1}]$ such that $V_1 \subset V_3$.

Proposition 4.20. *For a τ- or ρ-conic q, $\rho_{\widetilde{\mathscr{Y}}}([q])$ is the point corresponding to the quadric generated by lines which q parameterizes. For a σ-plane $\mathrm{P}_{\mathrm{V}_3}$, $\rho_{\widetilde{\mathscr{Y}}}([\mathrm{P}])$ is the point corresponding to the rank one quadric $2\mathbb{P}(V_3)$. In particular, the exceptional locus $F_{\widetilde{\mathscr{Y}}}$ consists of the points corresponding to τ- or ρ-conics of rank at most two or σ-planes, and the image of $F_{\widetilde{\mathscr{Y}}}$ coincides with $G_{\mathscr{Y}}$.*

Proof. We have described τ-conics and σ-planes in Examples 4.4 and 4.5 and Appendix A. The assertions for τ-conics and σ-planes follow from their descriptions and direct computations based on the results in Appendix B. For ρ-conics, the assertion follows from the isomorphism $F_\rho \simeq \widetilde{\mathrm{S}}_3$ as in the proof of Proposition 4.18. □

4.7. **Divisorial contraction $\rho_{\widetilde{\mathscr{Y}}}\colon \widetilde{\mathscr{Y}} \to \mathscr{Y}$ for $n \geq 4$ $(\dim V \geq 5)$.** Recall that we have the morphisms

$$\mathscr{Y}_3 \to \mathrm{G}(n-3, V),\ \ \mathscr{Y}_2 \to \mathrm{G}(n-3, V)\ \text{ and }\ \mathscr{U} \to \mathrm{G}(n-3, V)$$

from Definition 4.17 and (2.1) with $\mathscr{U} := \widetilde{\mathrm{S}}_4$. In this subsection, we consider the relative setting over $\mathrm{G}(n-3, V)$ for $n \geq 4$. Thus, for example, the geometry of $\mathscr{Y}_2$ is considered as the family of the blow-ups of $\mathrm{G}(3, \wedge^2(V/V_{n-3}))$ along $\mathscr{P}_\rho|_{[V_{n-3}]} = v_2(\mathbb{P}(V/V_{n-3}))$ for $[V_{n-2}] \in \mathrm{G}(n-3, V)$. The results in the preceding subsection apply to each member of the family with the 4-dimensional vector space V/V_{n-3}.

Lemma 4.21. *There exists a morphism $\mathscr{Y}_2 \to \mathscr{U}$ defined over $\mathrm{G}(n-3, V)$.*

Proof. Denote by $\mathscr{Y}_2|_{[V_{n-3}]}, \mathscr{Y}_3|_{[V_{n-3}]}, \mathscr{U}|_{[V_{n-3}]}$ the restrictions to the fibers over $[V_{n-3}] \in \mathrm{G}(n-3, V)$. Then $\mathscr{Y}_2|_{[V_{n-3}]}$ is the blow-up of $\mathscr{Y}_3|_{[V_{n-3}]} = \mathrm{G}(3, \wedge^2(V/V_{n-3}))$, as described above, and $\mathscr{U}|_{[V_{n-3}]} = \mathbb{P}(\mathsf{S}^2(V/V_{n-3})^*)$. The claimed morphism is the one described in Proposition 4.18 (1). □

Proposition 4.22. (1) *There exists an extremal divisorial contraction $\tilde{\rho}_{\mathscr{Y}_2}\colon \mathscr{Y}_2 \to \widetilde{\mathscr{Y}}$ which is the blow-up along G_ρ with the exceptional divisor F_ρ. Any fiber of $F_\rho \to G_\rho$ is a copy of $\mathbb{P}^{n-3}$ and is mapped to a fiber of $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$ isomorphically.*

(2) *There exists an extremal divisorial contraction $\rho_{\widetilde{\mathscr{Y}}}\colon \widetilde{\mathscr{Y}} \to \mathscr{Y}$.*

Proof. We reproduce here a part of the diagram (2.12):

$$\begin{tikzcd} \mathscr{Y}_{\mathscr{U}} = \widetilde{\mathrm{T}}_4 \arrow[r, "\rho_{\widetilde{\mathrm{T}}_4}"] \arrow[d, "p_{\widetilde{\mathrm{T}}_4}"'] & \mathscr{U} = \widetilde{\mathrm{S}}_4 \arrow[d, "\pi_{\widetilde{\mathrm{S}}_4}"'] \\ \mathscr{Y} = \mathrm{T}_4 \arrow[r, "\rho_{\mathrm{T}_4}"'] & \mathscr{H} = \mathrm{S}_4. \end{tikzcd} \tag{4.9}$$

By construction, we see that $\rho_{\widetilde{\mathrm{T}}_4} : \mathscr{Y}_{\mathscr{U}} \to \mathscr{U}$ is the family over $\mathrm{G}(n-3, V)$ of the double covers $\mathrm{T}_4 \to \mathrm{S}_4$ for 4-dimensional vector spaces V/V_{n-3}.

Consider the Stein factorization of the morphism $\mathscr{Y}_2 \to \mathscr{U}$. By the uniqueness of finite double cover, it is given by $\mathscr{Y}_2 \to \mathscr{Y}_{\mathscr{U}} \to \mathscr{U}$. Then the induced morphism $\mathscr{Y}_2 \to \mathscr{Y}_{\mathscr{U}}$ is the family over $\mathrm{G}(n-3, V)$ of the divisorial contraction described in Proposition 4.18 (2) (for 4-dimensional vector spaces V/V_{n-3}). In particular, a birational morphism $\mathscr{Y}_2 \to \mathscr{Y}$ is induced. By Proposition 4.11 and the definition of $\mathscr{Y}_2$, a birational morphism $\mathscr{Y}_2 \to \overline{\mathscr{Y}}'$ is also induced. Therefore we obtain a map $\mathscr{Y}_2 \to \overline{\mathscr{Y}}' \times \mathscr{Y}$. Let $\widetilde{\mathscr{Y}'}$ be the normalization of the image of this map. We will show that $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}'}$ is non-trivial. Let Q be a quadric in $\mathbb{P}(V)$ of rank 3 and $\mathbb{P}(V_{n-2})$ its singular locus. By Proposition 4.20, the fiber Γ of $\mathscr{Y}_2 \to \mathscr{Y}$ over $[Q]$ is isomorphic to $\mathrm{G}(n-3, V_{n-2})$. By Proposition 4.11 and the definition of $\mathscr{Y}_2$, Γ is also contracted by $\mathscr{Y}_2 \to \overline{\mathscr{Y}}'$. Therefore $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}'}$ is non-trivial. $\widetilde{\mathscr{Y}'}$ can not be isomorphic to $\overline{\mathscr{Y}}'$ nor $\mathscr{Y}$ since $\rho(\overline{\mathscr{Y}}') = \rho(\mathscr{Y}) = 1$ and $\overline{\mathscr{Y}}' \not\simeq \mathscr{Y}$. Therefore $\widetilde{\mathscr{Y}'} \to \overline{\mathscr{Y}}'$ is a small birational morphism. By the uniqueness of the flip (cf. [18]), we see that $\widetilde{\mathscr{Y}'} \simeq \widetilde{\mathscr{Y}}$ or $\mathscr{Y}_3$. There does not exist, however, a contraction $\mathscr{Y}_3 \to \mathscr{Y}$ since $\rho(\mathscr{Y}_3) = 2$ and there are two non-trivial contractions $\mathscr{Y}_3 \to \mathrm{G}(n-3, V)$ and $\mathscr{Y}_3 \to \overline{\mathscr{Y}}'$. Therefore we must have $\widetilde{\mathscr{Y}'} \simeq \widetilde{\mathscr{Y}}$. Now extending (4.9), we have

$$\begin{tikzcd} \mathscr{Y}_2 \arrow[r, "/\mathrm{G}(n-3,V)"] \arrow[d] & \mathscr{Y}_{\mathscr{U}} = \widetilde{\mathrm{T}}_4 \arrow[r, "/\mathrm{G}(n-3,V)", "\rho_{\widetilde{\mathrm{T}}_4}"'] \arrow[d, "p_{\widetilde{\mathrm{T}}_4}"'] & \mathscr{U} = \widetilde{\mathrm{S}}_4 \arrow[d, "\pi_{\widetilde{\mathrm{S}}_4}"'] \\ \widetilde{\mathscr{Y}} \arrow[r, "\rho_{\widetilde{\mathscr{Y}}}"'] & \mathscr{Y} = \mathrm{T}_4 \arrow[r, "\rho_{\mathrm{T}_4}"'] & \mathscr{H} = \mathrm{S}_4. \end{tikzcd} \tag{4.10}$$

Note that $\mathscr{Y}_2 \to \mathscr{Y}_{\mathscr{U}}$ and $\mathscr{Y}_{\mathscr{U}} \to \mathscr{Y}$ are divisorial contractions. Moreover, $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}}$ is also a divisorial contraction contracting F_ρ to G_ρ. Therefore $\widetilde{\mathscr{Y}} \to \mathscr{Y}$ is a divisorial contraction, and moreover its exceptional divisor $F_{\widetilde{\mathscr{Y}}}$ is the image of the exceptional divisor of $\mathscr{Y}_2 \to \mathscr{Y}_{\mathscr{U}}$.

Finally we show that $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}}$ is the blow-up of G_ρ. This morphism is given by forgetting the markings by $[V_{n-3}]$ in $\mathrm{G}(n-3, V)$. But, since $G_\rho \simeq \mathbb{P}(\mathsf{S}^2 \mathscr{Q}_\rho^*)$ (see Proposition 4.14), the markings by $[V_{n-2}]$ in $\mathrm{G}(n-2, V)$ are retained. Therefore the fiber of $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}}$ over a point $(q, [V_{n-2}])$ in $\mathbb{P}(\mathsf{S}^2 \mathscr{Q}_\rho^*)$ is isomorphic to $\mathrm{G}(n-3, V_{n-2}) \simeq \mathbb{P}^{n-3}$. We may conclude that $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}}$ is the blow-up of G_ρ by the same argument as in the proof of Proposition 4.15. □

Remark 4.23. In a similar way to the proof of Proposition 4.22 (1), we can show that $\mathscr{Y}_0 \to \widetilde{\mathscr{Y}}$ is the blow-up along G_σ.

By Propositions 4.20 and 4.22, we have the following:

Proposition 4.24. *For a τ- or ρ-conic q, $\rho_{\widetilde{\mathscr{Y}}}([q])$ is the point corresponding to the quadric generated by $\mathbb{P}(V_{n-1})$'s which q parameterizes. For a σ-plane $\mathrm{P}_{\mathrm{V_{n-3}V_n}}$, $\rho_{\widetilde{\mathscr{Y}}}([\mathrm{P}_{\mathrm{V_{n-3}V_n}}])$ is the point corresponding to the rank one quadric $2\mathbb{P}(V_n)$. In particular, the exceptional locus $F_{\widetilde{\mathscr{Y}}}$ consists of the points corresponding to τ- or ρ-conics of rank at most two or σ-planes, and the image of $F_{\widetilde{\mathscr{Y}}}$ coincides with $G_{\mathscr{Y}}$.*

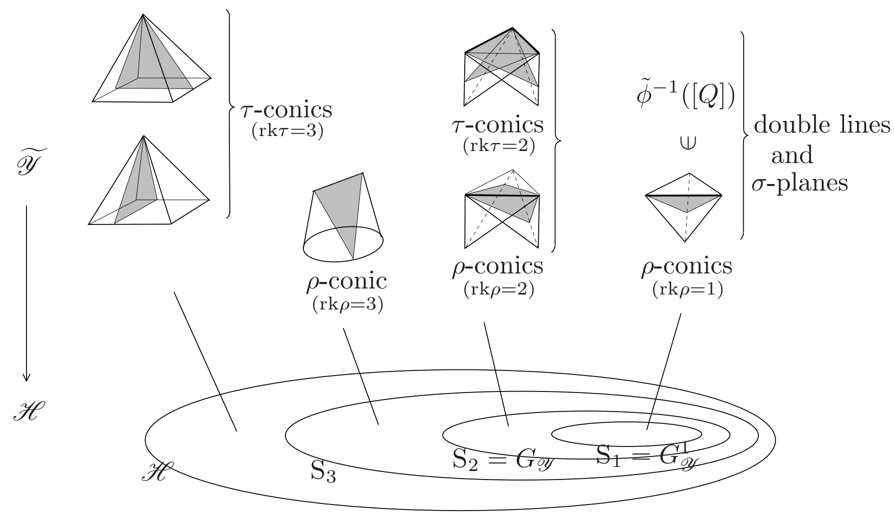

Fig.2. The fibers of $\tilde{\phi} = \rho_{\mathbf{T}_4} \circ \rho_{\widetilde{\mathscr{Y}}} \colon \widetilde{\mathscr{Y}} \to \mathscr{H}$ when $n = 4$.

5. Geometry of $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ and flattening

In this section, we determine the structure of $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ and construct its flattening.

5.1. **Birational model $F^{(1)}/\mathbb{Z}_2$ of $F_{\widetilde{\mathscr{Y}}}$.** From the description of the conics of rank two in Example 4.5 and Proposition 4.24, we introduce the following $\mathbb{Z}_2$-subvariety $F^{(1)}$ of $\mathrm{F}(n-2, n, V)^{\times 2}$ to study the exceptional locus $F_{\widetilde{\mathscr{Y}}} \subset \widetilde{\mathscr{Y}}$:

$$F^{(1)} := \left\{ ([V_{n-2}], [V'_{n-2}]; [V_n], [V'_n]) \;\middle|\; \begin{array}{c} V_{n-2}, V'_{n-2} \subset V_n \cap V'_n \\ \dim(V_{n-2} \cap V'_{n-2}) \geq n-3 \end{array} \right\}, \tag{5.1}$$

where $\mathbb{Z}_2$ acts by the simultaneous exchanges $V_{n-2} \leftrightarrow V'_{n-2}$ and $V_n \leftrightarrow V'_n$. We set

$$\widehat{G} := \mathbb{P}(V^*) \times \mathbb{P}(V^*),\ \Delta_G := \text{the diagonal of } \widehat{G},$$

and note that the natural projection $F^{(1)} \to \widehat{G}$ is a $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$-fibration outside Δ_G. Let $\overset{\circ}{F}{}^{(1)}$ be the following open subset of $F^{(1)}$:

$$\overset{\circ}{F}{}^{(1)} := \left\{ ([V_{n-2}], [V'_{n-2}]; [V_n], [V'_n]) \;\middle|\; V_n \neq V'_n \right\} \subset F^{(1)}. \tag{5.2}$$

Proposition 5.1. *The natural map $\overset{\circ}{F}{}^{(1)}/\mathbb{Z}_2 \to (\widehat{G} \setminus \Delta_G)/\mathbb{Z}_2$ is isomorphic to $F_{\widetilde{\mathscr{Y}}} \setminus \rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}}) \to G_{\mathscr{Y}} \setminus G^1_{\mathscr{Y}}$. In particular, $F_{\widetilde{\mathscr{Y}}} \setminus \rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}}) \to G_{\mathscr{Y}} \setminus G^1_{\mathscr{Y}}$ is a $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$-fibration.*

Proof. First note that $\widehat{G}/\mathbb{Z}_2 \simeq G_{\mathscr{Y}}$, $\Delta_G/\mathbb{Z}_2 \simeq G^1_{\mathscr{Y}}$ and hence $(\widehat{G} \setminus \Delta_G)/\mathbb{Z}_2 \simeq G_{\mathscr{Y}} \setminus G^1_{\mathscr{Y}}$.

Let us note that $\overset{\circ}{F}{}^{(1)}/\mathbb{Z}_2$ parameterizes line pairs in $\mathrm{G}(n-1, n+1)$ which are reducible conics of rank two and not on σ-planes (see Example 4.5 for explicit descriptions). Therefore we have the unique injective morphism $\overset{\circ}{F}{}^{(1)}/\mathbb{Z}_2 \to \mathscr{Y}_0$ which is induced by the universality of the Hilbert scheme $\mathscr{Y}_0$. By Proposition 4.24, the image of $\overset{\circ}{F}{}^{(1)}/\mathbb{Z}_2$ coincides with $F_{\widetilde{\mathscr{Y}}} \setminus \rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}})$, and the map $\overset{\circ}{F}{}^{(1)}/\mathbb{Z}_2 \to F_{\widetilde{\mathscr{Y}}} \setminus \rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}})$ induces the following commutative diagram:

$$\begin{array}{ccc} \overset{\circ}{F}{}^{(1)}/\mathbb{Z}_2 & \longrightarrow & F_{\widetilde{\mathscr{Y}}} \setminus \rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}}) \\ \downarrow & & \downarrow \\ (\widehat{G} \setminus \Delta_G)/\mathbb{Z}_2 & \xrightarrow{\simeq} & G_{\mathscr{Y}} \setminus G^1_{\mathscr{Y}}. \end{array}$$

Note that $F_{\widetilde{\mathscr{Y}}} \setminus \rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}})$ is normal. Indeed, $F_{\widetilde{\mathscr{Y}}}$ satisfies the S_2 condition since it is a divisor on a smooth variety. It also satisfies the R_1 condition since, by considering the $\mathrm{SL}(V)$-action, its singular locus is at most the locus of ρ-conics of rank two which is codimension $n-2 \geq 2$ in $F_{\widetilde{\mathscr{Y}}}$ if $n \geq 4$ (resp. it is smooth if $n = 3$ by Proposition 4.18 (5)). Hence $F_{\widetilde{\mathscr{Y}}} \setminus \rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}})$ is normal. Therefore the bijective morphism $\overset{\circ}{F}{}^{(1)}/\mathbb{Z}_2 \to F_{\widetilde{\mathscr{Y}}} \setminus \rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}})$ is an isomorphism by the Zariski main theorem.

Finally, the natural map $\overset{\circ}{F}{}^{(1)} \to \widehat{G}$ is obviously a $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$-fibration, and then so is $\overset{\circ}{F}{}^{(1)}/\mathbb{Z}_2 \to (\widehat{G} \setminus \Delta_G)/\mathbb{Z}_2$ since the $\mathbb{Z}_2$-action interchanges the fibers over (x, y) and (y, x) in $\widehat{G} \setminus \Delta_G$. □

The following corollary will be used in the companion paper [13].

Corollary 5.2. *It holds that*

$$K_{\widetilde{\mathscr{Y}}} = \rho^*_{\widetilde{\mathscr{Y}}} K_{\mathscr{Y}} + (n-2) F_{\widetilde{\mathscr{Y}}}. \tag{5.3}$$

Proof. Let a be the discrepancy of $F_{\widetilde{\mathscr{Y}}}$. We show $a = n-2$. Let $\Gamma \simeq \mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$ be a fiber of $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ outside the diagonal of $G_{\mathscr{Y}}$ and l a line in a ruling of $\Gamma \simeq \mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$. Since $K_\Gamma \cdot l = -(n-1)$ and $K_\Gamma = K_{F_{\widetilde{\mathscr{Y}}}}|_\Gamma = (a+1) F_{\widetilde{\mathscr{Y}}}|_\Gamma$, we have $(a+1) F_{\widetilde{\mathscr{Y}}} \cdot l = -(n-1)$. Therefore we have only to show $F_{\widetilde{\mathscr{Y}}} \cdot l = -1$. For this we take l so that $l \cap G_\rho \neq \emptyset$. Now we consider the diagram (4.9). Since $\Gamma \cap G_\rho$ is the diagonal by Proposition 5.1, the strict transform l' is a ruling of a fiber $\simeq \mathbb{P}^1 \times \mathbb{P}^1$ of $\mathscr{Y}_2 \to \mathscr{Y}_{\mathscr{U}}$. Therefore $F'_{\widetilde{\mathscr{Y}}} \cdot l' = -1$ where $F'_{\widetilde{\mathscr{Y}}}$ is the strict transform of $F_{\widetilde{\mathscr{Y}}}$. Since $G_\rho \not\subset F_{\widetilde{\mathscr{Y}}}$, we have $F_{\widetilde{\mathscr{Y}}} \cdot l = F'_{\widetilde{\mathscr{Y}}} \cdot l' = -1$ as desired. □

By Proposition 5.1, we have a birational map $F^{(1)}/\mathbb{Z}_2 \dashrightarrow F_{\widetilde{\mathscr{Y}}}$ extending the isomorphism $\overset{\circ}{F}{}^{(1)}/\mathbb{Z}_2 \simeq F_{\widetilde{\mathscr{Y}}} \setminus \rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}})$. In the sequel of this section, we will give an explicit description of this birational map using the minimal model theory, which leads to a precise description of $F_{\widetilde{\mathscr{Y}}}$. We summarize our description in the following diagram:

$$
\begin{tikzcd}
 & F^{(3)} \arrow[dl] \arrow[dr] & & \\
F^{(2)} \arrow[rr, dashed, "\text{(anti-)flip}"] \arrow[dr] \arrow[ddr, "\mathbb{P}^{n-2}\times\mathbb{P}^{n-2}\text{-fib.}"'] & & F^{(4)} \arrow[dl] \arrow[d, "\text{div. cont.}"] & \\
 & F^{(1)} \arrow[dr] & \widehat{F} \arrow[r, "\mathbb{Z}_2\text{-quot.}"'] \arrow[d] & F_{\widetilde{\mathscr{Y}}} \arrow[d, "\rho_{\widetilde{\mathscr{Y}}}|_{F_{\widetilde{\mathscr{Y}}}}"] \\
 & \widehat{G}' \arrow[r, "\text{diag.blow up}"'] & \widehat{G} \arrow[r, "\mathbb{Z}_2\text{-quot.}"] & G_{\mathscr{Y}}\,.
\end{tikzcd}
\tag{5.4}
$$

5.2. **Small resolution and flip.** First we determine the singularities of $F^{(1)}$.

Proposition 5.3. *$F^{(1)}$ is singular along the diagonal set*

$$\Delta_{F^{(1)}} := \{([V_{n-2}],[V_{n-2}];[V_n],[V_n]) \mid V_{n-2} \subset V_n\} \simeq \mathrm{F}(n-2,n,V) \subset F^{(1)}. \tag{5.5}$$

The singularity at each point on $\Delta_{F^{(1)}}$ is isomorphic to the cone over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{n-2}$.

Proof. Recall that $F^{(1)}$ is a subvariety of $\mathrm{F}(n-2,n,V)^{\times 2}$ and consider the first projection $F^{(1)} \to \mathrm{F}(n-2,n,V)$. Let Γ be a fiber of this projection over a point $([V_{n-2}];[V_n]) \in \mathrm{F}(n-2,n,V)$. We consider Γ as a subvariety of $\mathrm{F}(n-2,n,V)$ parameterizing $V'_{n-2} \subset V'_n$ such that $V'_{n-2} \subset V_n$, $V_{n-2} \subset V'_n$ and $\dim(V_{n-2} \cap V'_{n-2}) \geq n-3$. To describe Γ, we choose a basis $\{\mathbf{e}_1,\dots,\mathbf{e}_{n+1}\}$ of V so that $V_{n-2} = \langle \mathbf{e}_1,\dots,\mathbf{e}_{n-2}\rangle$ and $V_n = \langle \mathbf{e}_1,\dots,\mathbf{e}_n\rangle$. An $(n-2)$-dimensional subspace V'_{n-2} of V_n with $\dim(V_{n-2}\cap V'_{n-2}) \geq n-3$ is spanned by $n-3$ vectors in V_{n-2} and a vector $b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n$ in V_n. We arrange these vectors into an $(n-2)\times n$ matrix as

$$\begin{pmatrix} \quad A \quad & \mathbf{0} & \mathbf{0} \\ b_1 \dots b_{n-2} & b_{n-1} & b_n \end{pmatrix}, \tag{5.6}$$

where the row vectors of A represents the $n-3$ vectors in V_{n-2}. We denote by q_{ij} the Plücker coordinate of V'_{n-2} given by the $(n-2)\times(n-2)$ minors of (5.6) with the i- and j-th columns omitted. Denote by $x_1,\dots,x_{n+1}$, and $y_1,\dots,y_{n+1}$ the homogeneous coordinates of $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, respectively, associated to the basis $\{\mathbf{e}_1,\dots,\mathbf{e}_{n+1}\}$ and its dual basis. An n-dimensional subspace V'_n of V containing V_{n-2} is of the form $\{c_{n-1}x_{n-1} + c_nx_n + c_{n+1}x_{n+1} = 0\}$, where we consider $(0,\dots,0,c_{n-1},c_n,c_{n+1})$ as the coordinates of $[V'_n]$ in V^*. Therefore V'_n contains V'_{n-2} if and only if $c_{n-1}b_{n-1} + c_nb_n = 0$. From the above considerations, we can deduce that

$$\Gamma = \left\{ (q_{ij};y_1,\dots,y_{n+1}) \;\middle|\; \begin{array}{l} q_{ij} = 0 \text{ for } 1 \leq i,j \leq n-2, \\ \operatorname{rank} \begin{pmatrix} q_{1\,n} & q_{2\,n} & \cdots & q_{n-2\,n} & -y_n \\ q_{1\,n-1} & q_{2\,n-1} & \cdots & q_{n-2\,n-1} & y_{n-1} \end{pmatrix} \leq 1 \end{array} \right\}.$$

From this, it is easy to see the assertion. □

The cone over $\mathbb{P}^1 \times \mathbb{P}^{n-2}$ has exactly two small resolutions; one of which has a $\mathbb{P}^1$ as the exceptional set and another has a $\mathbb{P}^{n-2}$ as the exceptional set. Corresponding

to these, we have two small resolutions of $F^{(1)}$. One of them is given by the following variety $F^{(2)}$:

$$\begin{aligned} F^{(2)} &:= \mathrm{F}(n-2,n-1,n,V)\times_{\mathrm{G}(n-1,V)}\mathrm{F}(n-2,n-1,n,V)\\ &= \left\{([V_{n-2}],[V'_{n-2}];[V_{n-1}];[V_n],[V'_n]) \,\middle|\, V_{n-2},V'_{n-2}\subset V_{n-1}\subset V_n\cap V'_n\right\}. \end{aligned}$$

We set

$$\begin{aligned} \hat{G}' &:= \mathrm{F}(n-1,n,V)\times_{\mathrm{G}(n-1,V)}\mathrm{F}(n-1,n,V)\\ &= \{([V_{n-1}];[V_n],[V'_n]) \mid V_{n-1}\subset V_n\cap V'_n\}. \end{aligned}$$

$F^{(2)}$ has a $\mathbb{P}^{n-2}\times\mathbb{P}^{n-2}$-fibration $F^{(2)}\to\hat{G}'$. We note that there is a morphism $\widehat{G}'\to\widehat{G}=\mathbb{P}(V^*)\times\mathbb{P}(V^*)$ defined by $([V_{n-1}];[V_n],[V'_n])\mapsto([V_n],[V'_n])$, which is nothing but the blow-up of $\widehat{G}$ along the diagonal Δ_G.

Proposition 5.4. (1) *$F^{(2)}$ is smooth. The natural projection $F^{(2)}\to F^{(1)}$ is a small resolution with every non-trivial fiber γ being isomorphic to $\mathbb{P}^1$.*
(2) *The normal bundle $\mathcal{N}_{\gamma/F^{(2)}}$ of a non-trivial fiber γ of $F^{(2)}\to F^{(1)}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n-1}\oplus\mathcal{O}_{\mathbb{P}^1}^{\oplus 3n-4}$.*
(3) *There is another small resolution $F^{(4)}\to F^{(1)}$, whose non-trivial fiber is isomorphic to $\mathbb{P}^{n-2}$. $F^{(2)}$ and $F^{(4)}$ fit into the following diagram:*

$$\begin{tikzcd} & F^{(3)} \arrow[dl, "p"'] \arrow[dr] & \\ F^{(2)} \arrow[dr] & & F^{(4)} \arrow[dl] \\ & F^{(1)}, & \end{tikzcd} \tag{5.7}$$

where $p\colon F^{(3)}\to F^{(2)}$ is the blow-up along the exceptional locus of $F^{(2)}\to F^{(1)}$, and $F^{(3)}\to F^{(4)}$ is the contraction of the exceptional divisor of the blow-up $F^{(3)}\to F^{(2)}$ in another direction.

Proof. (1) $F^{(2)}$ is smooth since it has a $\mathbb{P}^{n-2}\times\mathbb{P}^{n-2}$-fibration over a smooth variety $\widehat{G}'$. We show that $F^{(2)}\to F^{(1)}$ is a small resolution. For a point

$$([V_{n-2}],[V'_{n-2}];[V_{n-1}];[V_n],[V'_n])\in F^{(2)},$$

$V_{n-1}=V_{n-2}+V'_{n-2}$ holds when $V_{n-2}\neq V'_{n-2}$, and also $V_{n-1}=V_n\cap V'_n$ when $V_n\neq V'_n$. Hence the morphism $F^{(2)}\to F^{(1)}$ is isomorphic outside the diagonal set $\Delta_{F^{(1)}}$. The fiber of $F^{(2)}\to F^{(1)}$ over a point $([V_{n-2}],[V_{n-2}];[V_n],[V_n])\in\Delta_{F^{(1)}}$ is

$$\{([V_{n-2}],[V_{n-2}];[V_{n-1}];[V_n],[V_n]) \mid [V_{n-1}]\in\mathrm{G}(n-1,V), V_{n-2}\subset V_{n-1}\subset V_n\}\simeq\mathbb{P}^1.$$

We calculate the dimension of the exceptional set of $F^{(2)}\to F^{(1)}$ as $\dim\Delta_{F^{(1)}}+1=3n-3$. Hence $F^{(2)}\to F^{(1)}$ is small since $\dim F^{(1)}=4n-4$.

(2) The two small resolutions of $F^{(1)}$ locally coincide with those of the cone over $\mathbb{P}^1\times\mathbb{P}^{n-3}$. Therefore the description of the normal bundle of γ follows by that of a non-trivial fiber of the small resolutions of the cone over $\mathbb{P}^1\times\mathbb{P}^{n-3}$.

(3) Let D be the p-exceptional divisor. Then any fiber of D is $\mathbb{P}^1\times\mathbb{P}^{n-2}$ by Proposition 5.3. Let $\gamma\simeq\mathbb{P}^1$ be a fiber of $F^{(2)}\to F^{(1)}$. Since $K_{F^{(2)}}\cdot\gamma=n-3$ by (2), we see that $p^*K_{F^{(2)}}+(n-3)D$ is nef and $(p^*K_{F^{(2)}}+(n-3)D)-K_{F^{(3)}}=-D$ is nef and big over $F^{(1)}$, $p^*K_{F^{(2)}}+(n-3)D$ is semi-ample over $F^{(1)}$ by the Kawamata-Shokurov base point free theorem. Since $p^*K_{F^{(2)}}+D$ is numerically trivial for

any fiber γ' of $\mathbb{P}^1 \times \mathbb{P}^{n-2} \to \mathbb{P}^{n-2}$, the birational morphism $F^{(3)} \to F^{(4)}$ over $F^{(1)}$ defined by a sufficiently high multiple of $p^*K_{F^{(2)}} + (n-3)D$ contracts γ'. Since $-K_{F^{(3)}} \cdot \gamma' = 1$ by (3), $F^{(4)}$ is smooth and $F^{(3)} \to F^{(4)}$ is the blow-up along the image of D (cf. the proof of Proposition 4.15 in case $n = 3$). $\square$

5.3. **Divisorial contraction.** Let $D^{(2)}$ be the inverse image in $F^{(2)}$ of the diagonal Δ_G of $\widehat{G}$, namely,

$$D^{(2)} := \mathrm{F}(n-2, n-1, n, V) \times_{\mathrm{F}(n-1,n,V)} \mathrm{F}(n-2, n-1, n, V).$$

We denote by $D^{(1)}$ the image on $F^{(1)}$ of $D^{(2)}$. It is easy to verify the following properties:

Lemma 5.5. (1) *$D^{(2)}$ is a prime divisor of $F^{(2)}$.*
(2) *The projection $D^{(2)} \to \mathrm{F}(n-1, n, V)$ is a $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$-fibration.*
(3) *All the non-trivial fibers of $F^{(2)} \to F^{(1)}$ are contained in $D^{(2)}$, namely, they coincide with the fibers of $D^{(2)} \to D^{(1)}$. Therefore $D^{(2)} \to D^{(1)}$ is birational with any non-trivial fiber being a copy of $\mathbb{P}^1$.*

Now we set

$$\begin{aligned} D^{(4)} &:= \mathrm{F}(n-3, n-2, n, V) \times_{\mathrm{F}(n-3,n,V)} \mathrm{F}(n-3, n-2, n, V) \\ &= \left\{ ([V_{n-3}]; [V_{n-2}], [V'_{n-2}]; [V_n], [V_n]) \;\middle|\; \begin{array}{l} V_{n-3} \subset V_{n-2} \cap V'_{n-2}, \\ V_{n-2}, V'_{n-2} \subset V_n \end{array} \right\}. \end{aligned} \tag{5.8}$$

Then we can deduce easily the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
D^{(2)} \arrow[rr, dashed] \arrow[d, "\mathbb{P}^{n-2}\times\mathbb{P}^{n-2}\text{-fib.}"'] \arrow[dr] & & D^{(4)} \arrow[d, "\mathbb{P}^{2}\times\mathbb{P}^{2}\text{-fib.}"] \arrow[dl] \\
\mathrm{F}(n-1, n, V) \arrow[dr] & D^{(1)} \arrow[d] & \mathrm{F}(n-3, n, V) \arrow[dl] \\
& \Delta_G, &
\end{tikzcd} \tag{5.9}$$

where $D^{(4)} \to D^{(1)}$ is birational with any non-trivial fiber being a copy of $\mathbb{P}^{n-3}$.

Lemma 5.6. *$D^{(4)}$ is the strict transform on $F^{(4)}$ of $D^{(2)}$, and the diagram* (5.9) *follows from the restriction of* (5.7).

Proof. In a similar way to the case of $F^{(1)}$, we may show that $D^{(1)}$ is singular along $\Delta_{F^{(1)}}$, and the singularity at each point on $\Delta_{F^{(1)}}$ is isomorphic to the cone over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{n-3}$ if $n \geq 4$ ($D^{(1)}$ is smooth if $n = 3$). Moreover, by restricting (5.7) to $D^{(1)}$ and its strict transforms, we have a similar diagram for $D^{(1)}$. In particular, the restriction of (5.7) gives two small resolutions of $D^{(1)}$ if $n \geq 4$ (for $n = 3$, the restriction of $F^{(2)} \to F^{(1)}$ is the blow-up along $\Delta_{F^{(1)}}$, and the restriction of $F^{(4)} \to F^{(1)}$ is an isomorphism). Let us define

$$\begin{aligned} D^{(3)} &:= \mathrm{F}(n{-}3, n{-}2, n{-}1, n, V) \times_{\mathrm{F}(n-3,n-1,n,V)} \mathrm{F}(n{-}3, n{-}2, n{-}1, n, V) \\ &= \left\{ ([V_{n-3}]; [V_{n-2}], [V'_{n-2}]; [V_{n-1}]; [V_n], [V_n]) \;\middle|\; \begin{array}{l} V_{n-3} \subset V_{n-2}, \\ V'_{n-2} \subset V_{n-1} \subset V_n. \end{array} \right\} \end{aligned} \tag{5.10}$$

Then $D^{(1)},\dots,D^{(4)}$ fit into the following diagram with the natural projections:

$$\begin{tikzcd} & D^{(3)} \arrow[dl] \arrow[dr] & \\ D^{(2)} \arrow[dr] & & D^{(4)} \arrow[dl] \\ & D^{(1)}. & \end{tikzcd} \tag{5.11}$$

By construction, it is easy to see that $D^{(2)}\to D^{(1)}$ and $D^{(4)}\to D^{(1)}$ are two small resolutions of $D^{(1)}$ if $n\geq 4$ (for $n=3$, $D^{(2)}\to D^{(1)}$ is the blow-up along $\Delta_{F^{(1)}}$ and $D^{(4)}\to D^{(1)}$ is an isomorphism). Therefore the diagram (5.11) coincides with the restriction of (5.7) considered above, and hence the assertions follow. $\square$

Proposition 5.7. *There exists a divisorial contraction $F^{(4)}\to \widehat{F}$ over $\widehat{G}$ which contracts the strict transform $D^{(4)}$ of $D^{(1)}$ to the locus isomorphic to the flag variety* $\mathrm{F}(n-3,n,V)$. *The discrepancy of $D^{(4)}$ is two.*

Proof. Let $\Delta'_{\mathbb{P}}$ be the inverse image in $\widehat{G}'$ of Δ_G. Note that $\Delta'_{\mathbb{P}}\simeq \mathrm{F}(n-1,n,V)$. Let Γ be a fiber of the $\mathbb{P}^{n-2}\times\mathbb{P}^{n-2}$-fibration $D^{(2)}\to\Delta'_{\mathbb{P}}$. Then Γ intersects the flipping locus of $F^{(2)}\dashrightarrow F^{(4)}$ along the diagonal transversally. Take a line $r\subset\mathbb{P}^{n-2}\times\mathbb{P}^{n-2}$ which is contained in a fiber of the second projection $\Gamma\to\mathbb{P}^{n-2}$ and intersects the flipping locus. r is of the form with some fixed $V_{n-3}\subset V'_{n-2}\subset V_{n-1}\subset V_n$ and moving V_{n-2} as follows:

$$r:=\{([V_{n-2}],[V'_{n-2}];[V_{n-1}];[V_n],[V_n])\mid V_{n-3}\subset V_{n-2}\subset V_{n-1}\}.$$

Then its strict transform r' on $D^{(4)}$ is contracted by the morphism $D^{(4)}\to\mathrm{F}(n-3,n,V)$. Since $F^{(2)}\to\widehat{G}'$ is a $\mathbb{P}^{n-2}\times\mathbb{P}^{n-2}$-fibration and $D^{(2)}$ is the pull-back of $\Delta'_{\mathbb{P}}$, we see that $K_{F^{(2)}}\cdot r=-(n-1)$ and $D^{(2)}\cdot r=0$. By the standard calculations of the changes of the intersection numbers by the flip, we have $K_{F^{(4)}}\cdot r'=-(n-1)+(n-3)=-2$ and $D^{(4)}\cdot r'=0-1=-1$. These equalities of the intersection numbers still hold for any line in a ruling of a fiber of $D^{(4)}\to\mathrm{F}(n-3,n,V)$.

We show $-K_{F^{(4)}}+2D^{(4)}$ is relatively nef over $\widehat{G}$. Let γ be a curve on $F^{(4)}$ which is contracted to a point t on $\widehat{G}$. If $t\notin\Delta_G$, then $(-K_{F^{(4)}}+2D^{(4)})\cdot\gamma>0$ since $D^{(4)}\cap\gamma=\emptyset$ and $F^{(4)}\to\widehat{G}$ is a $\mathbb{P}^{n-2}\times\mathbb{P}^{n-2}$ fibration outside Δ_G. If $t\in\Delta_G$ and γ is an exceptional curve of $F^{(4)}\to F^{(1)}$, then $(-K_{F^{(4)}}+2D^{(4)})\cdot\gamma>0$ since $-K_{F^{(4)}}\cdot\gamma>0$ and $D^{(4)}\cdot\gamma>0$. In the remaining cases, $t\in\Delta_G$ and $\gamma\subset D^{(4)}$. Therefore we have only to consider the relative nefness of $(-K_{F^{(4)}}+2D^{(4)})|_{D^{(4)}}$ over Δ_G. Now we take as γ any line in a ruling of a fiber of $D^{(4)}\to\mathrm{F}(n-3,n,V)$. As we see in the first paragraph, $(-K_{F^{(4)}}+2D^{(4)})\cdot\gamma=0$. Therefore $(-K_{F^{(4)}}+2D^{(4)})|_{D^{(4)}}$ is the pull-back of some divisor D_F on $\mathrm{F}(n-3,n,V)$. It suffices to show D_F is relatively nef over Δ_G, which is true since an exceptional curve of $D^{(4)}\to D^{(1)}$ is positive for $(-K_{F^{(4)}}+2D^{(4)})|_{D^{(4)}}$ as above and is mapped to a curve on a fiber of $\mathrm{F}(n-3,n,V)\to\Delta_G$. Therefore $-K_{F^{(4)}}+2D^{(4)}$ is relatively nef over $\widehat{G}$.

Moreover, by this argument, we see that $(-K_{F^{(4)}}+2D^{(4)})^{\perp}\cap\overline{\mathrm{NE}}(F^{(4)}/\widehat{G})$ is generated by the numerical class of the curves on fibers of $D^{(4)}\to\mathrm{F}(n-3,n,V)$. In particular, $(-K_{F^{(4)}}+2D^{(4)})^{\perp}\cap\overline{\mathrm{NE}}(F^{(4)}/\widehat{G})\subset(K_{F^{(4)}})^{<0}$. Therefore, by Mori theory, there exists a contraction associated to this extremal face, which is nothing but the divisorial contraction contracting $D^{(4)}$ to $\mathrm{F}(n-3,n,V)$ such that $-K_{F^{(4)}}+2D^{(4)}$ is the pull-back of $-K_{\widehat{F}}$. Thus the discrepancy of $D^{(4)}$ is two. $\square$

5.4. $\mathbb{Z}_2$**-quotient.** All the above constructions are $\mathbb{Z}_2$-equivariant, hence we can take $\mathbb{Z}_2$-quotient $\widehat{F}/\mathbb{Z}_2$. Comparing the morphisms $a\colon F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ and $b\colon \widehat{F}/\mathbb{Z}_2 \to G_{\mathscr{Y}}$, we obtain

Proposition 5.8. *$\widehat{F}/\mathbb{Z}_2 \simeq F_{\widetilde{\mathscr{Y}}}$ over $G_{\mathscr{Y}}$.*

Lemma 5.9. *The fiber of $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ at any point of $G^1_{\mathscr{Y}}$ is of dimension at most $3n-6$. In particular, codimension of the inverse image in $F_{\widetilde{\mathscr{Y}}}$ of $G^1_{\mathscr{Y}}$ is at least two.*

Proof. We consider the diagram (4.9). By Proposition 4.18 (5), the fiber of $\mathscr{Y}_2 \to \mathscr{Y}_{\mathscr{U}}$ over a rank one point in a fiber of $\mathscr{Y}_{\mathscr{U}} \to \mathrm{G}(n-3, V)$ is isomorphic to $\mathbb{P}(1^3, 2)$. The fiber of $\mathscr{Y}_{\mathscr{U}} \to \mathscr{Y}$ over a rank one point is isomorphic to that of $\mathscr{U} \to \mathscr{H}$ over a rank one point $[2V_n] \in \mathrm{S}_1$, and hence is a copy of $\mathrm{G}(n-3, V_n)$. Therefore, the fiber of $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ at any point of $G^1_{\mathscr{Y}}$ is of dimension at most $3+3(n-3) = 3n-6$. □

Proof of Proposition 5.8**.** Note that the morphisms a and b are isomorphic outside $G^1_{\mathscr{Y}}$ by Proposition 5.1. Therefore, by [25, Lem. 5.5] for example, it suffices to check the following properties:

(1) The inverse images of $G^1_{\mathscr{Y}}$ by the morphisms a and b are of codimension at least two.
(2) Both $F_{\widetilde{\mathscr{Y}}}$ and $\widehat{F}/\mathbb{Z}_2$ are normal.
(3) $-K_{F_{\widetilde{\mathscr{Y}}}}$ and $-K_{\widehat{F}/\mathbb{Z}_2}$ are $\mathbb{Q}$-Cartier.
(4) $-K_{F_{\widetilde{\mathscr{Y}}}}$ is a-ample and $-K_{\widehat{F}/\mathbb{Z}_2}$ is b-ample.

We show these in order.
(1) The inverse image of $G^1_{\mathscr{Y}}$ by the morphism a has codimension at least two in $F_{\widetilde{\mathscr{Y}}}$ by Lemma 5.9 and the inverse image of $G^1_{\mathscr{Y}}$ by the morphism b has codimension two in $\widehat{F}/\mathbb{Z}_2$ by the construction of $\widehat{F}/\mathbb{Z}_2$.
(2) The variety $F_{\widetilde{\mathscr{Y}}}$ is normal. Indeed, it satisfies the S_2 condition since it is a Cartier divisor on a smooth variety. It satisfies the R_1 condition since it is a $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$-fibration outside the locus of codimension at least two by Proposition 5.1 and Lemma 5.9. We see that the variety $\widehat{F}/\mathbb{Z}_2$ is normal by its explicit construction as above.
(3), (4) The divisor $-K_{F_{\widetilde{\mathscr{Y}}}}$ is $\mathbb{Q}$-Cartier since $F_{\widetilde{\mathscr{Y}}}$ is a divisor on the smooth variety $\widetilde{\mathscr{Y}}$. We see that $-K_{F_{\widetilde{\mathscr{Y}}}}$ is a-ample since the relative Picard number $\rho(\widetilde{\mathscr{Y}}/\mathscr{Y})$ is one and a is generically a $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$-fibration.

Arguments for the morphism b are similar. Let us first show that $-K_{\widehat{F}/\mathbb{Z}_2}$ is $\mathbb{Q}$-Cartier. Indeed, by Lemma 5.7, the discrepancy of $D^{(4)}$ is two. Then, by the Kawamata-Shokurov base point free theorem, $-K_{F^{(4)}} - 2D^{(4)}$ is the pull-back of a Cartier divisor on $\widehat{F}$, which turns out to be the anti-canonical divisor $-K_{\widehat{F}}$. Thus $-K_{\widehat{F}/\mathbb{Z}_2}$ is $\mathbb{Q}$-Cartier.

To show $-K_{\widehat{F}/\mathbb{Z}_2}$ is b-ample, it suffices to see the relative Picard number $\rho((\widehat{F}/\mathbb{Z}_2)/G_{\mathscr{Y}})$ is one because b is generically a $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$-fibration. We compute $\rho((\widehat{F}/\mathbb{Z}_2)/G_{\mathscr{Y}})$ using the above construction. The relative Picard number $\rho(F^{(2)}/\widehat{G}')$ is two since $F^{(2)} \to \widehat{G}'$ is a $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$-fibration and it is easy to see that it is the composite of two $\mathbb{P}^{n-2}$-fibrations. Moreover we have $\rho^{\mathbb{Z}_2}(F^{(2)}/\widehat{G}') = 1$ since rulings in two directions of a fiber $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$ of $F^{(2)} \to \widehat{G}'$ are exchanged by the $\mathbb{Z}_2$-action. Therefore $\rho^{\mathbb{Z}_2}(F^{(2)}) = 3$ since $\rho^{\mathbb{Z}_2}(\widehat{G}') = 2$. It holds that $\rho^{\mathbb{Z}_2}(F^{(4)}) = 3$ since the

flip preserves the Picard number and the flip is $\mathbb{Z}_2$-equivariant. Since a divisorial contraction drops the Picard number at least by one, we have $\rho^{\mathbb{Z}_2}(\widehat{F}) \leq 2$. Now we see that $\rho((\widehat{F}/\mathbb{Z}_2)/G_{\mathscr{Y}})$ is one since $\rho(G_{\mathscr{Y}}) = 1$ and the morphism $\widehat{F}/\mathbb{Z}_2 \to G_{\mathscr{Y}}$ is non-trivial. Therefore we conclude $-K_{\widehat{F}/\mathbb{Z}_2}$ is b-ample. $\square$

5.5. **Flattening** $F^{(3)} \to \widehat{G}'$ **.** We describe the fibers of $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ in the diagram (5.4).

Proposition 5.10. *There is a birational morphism* $\mathbb{P}(\mathcal{O}_{\mathrm{G}(n-2,V_n)} \oplus \mathcal{U}^*_{\mathrm{G}(n-2,V_n)}(1)) \to \rho^{-1}_{\widetilde{\mathscr{Y}}}([V_n])$ *which contracts the divisor* $\mathbb{P}(\mathcal{U}^*_{V_n}(1))$ *to* $\mathrm{G}(n-3, V_n)$*, where* $\mathcal{U}_{\mathrm{G}(n-2,V_n)}$ *is the universal subbundle of the Grassmannian* $\mathrm{G}(n-2, V_n)$*.*

Proof. Since the fiber under consideration is contained in the branched locus of $\widehat{F} \to F_{\widetilde{\mathscr{Y}}}$, we have only to describe the fiber Γ of $\widehat{F} \to \widehat{G}$ over $[V_n]$, where we consider $[V_n]$ is a point of the diagonal of $\widehat{G}$. Let Γ' be the restriction over $[V_n]$ of the exceptional locus of $F^{(4)} \to F^{(1)}$. Then the fiber Γ is nothing but the image of Γ' under the divisorial contraction $F^{(4)} \to \hat{F}$. Since the fiber of $\Delta_{F^{(1)}} \to \widehat{G}$ over $[V_n]$ is $\mathrm{G}(n-2, V_n)$, the variety Γ' is a $\mathbb{P}^{n-2}$-bundle over $\mathrm{G}(n-2, V_n)$. By the definition of $D^{(4)}$, we see that $D^{(4)}|_{\Gamma'} = \mathrm{F}(n-3, n-2, V_n)$, which is isomorphic to $\mathbb{P}(\mathcal{U}^*_{\mathrm{G}(n-2,V_n)}(-1))$. Therefore we may write $\Gamma' = \mathbb{P}(\mathcal{A}^*)$, where $\mathcal{A}$ is the locally free sheaf of rank $n-2$ on $\mathrm{G}(n-2, V_n)$ with a surjection $\mathcal{A} \to \mathcal{U}_{\mathrm{G}(n-2,V_n)}(1)$. Now we show the kernel of $\mathcal{A} \to \mathcal{U}_{\mathrm{G}(n-2,V_n)}(1)$ is $\mathcal{O}_{\mathrm{G}(n-2,V_n)}(2)$. Note that the image of $\mathrm{F}(n-3, n-2, V_n)$ by the divisorial contraction $F^{(4)} \to \widehat{F}$ is $\mathrm{G}(n-3, V_n)$. Therefore, since the discrepancy of $D^{(4)}$ for $F^{(4)} \to \widehat{F}$ is two, and $\mathcal{O}_{\mathbb{P}(\mathcal{U}^*_{\mathrm{G}(n-2,V_n)}(-1))}(1)$ is the pull-back of $\mathcal{O}_{\mathrm{G}(n-3,V_n)}(1)$, we see that $D^{(4)}|_{\Gamma'} = H_{\mathbb{P}(\mathcal{A}^*)} - 2L$, where L is the pull-back of $\mathcal{O}_{\mathrm{G}(n-2,V_n)}(1)$. Thus the kernel of $\mathcal{A} \to \mathcal{U}_{\mathrm{G}(n-2,V_n)}(1)$ is $\mathcal{O}_{\mathrm{G}(n-2,V_n)}(2)$. Since the exact sequence $0 \to \mathcal{O}_{\mathrm{G}(n-2,V_n)}(2) \to \mathcal{A} \to \mathcal{U}_{\mathrm{G}(n-2,V_n)}(1) \to 0$ splits, we have $\mathcal{A}^* \simeq \mathcal{O}_{\mathrm{G}(n-2,V_n)}(-2) \oplus \mathcal{U}^*_{\mathrm{G}(n-2,V_n)}(-1) \simeq (\mathcal{O}_{\mathrm{G}(n-2,V_n)} \oplus \mathcal{U}^*_{\mathrm{G}(n-2,V_n)}(1)) \otimes \mathcal{O}_{\mathrm{G}(n-2,V_n)}(-2)$. $\square$

We have obtained the following diagram:

$$
\begin{tikzcd}
F^{(3)} \arrow[r] \arrow[d] & F^{(4)} \arrow[r, "\text{div. cont.}"] \arrow[d] & \widehat{F} \arrow[r, "\mathbb{Z}_2\text{-quot.}"] \arrow[d] & F_{\widetilde{\mathscr{Y}}} \arrow[d] \\
\widehat{G}' \arrow[r] & \widehat{G} \arrow[r, equal] & \widehat{G} \arrow[r, "\mathbb{Z}_2\text{-quot.}"] & G_{\mathscr{Y}}
\end{tikzcd}
\tag{5.12}
$$

We show that $F^{(3)} \to \widehat{G}'$ gives a flattening of the fibration $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$.

Proposition 5.11. $F^{(3)} \to \widehat{G}'$ *is flat. More precisely, the fiber* $Fib^{(3)}(V_{n-1}, V_n, V_n')$ *of* $F^{(3)} \to \widehat{G}'$ *over a point* $([V_{n-1}]; [V_n], [V_n'])$ *have the following descriptions:*

(1) $Fib^{(3)}(V_{n-1}, V_n, V_n') \simeq \mathbb{P}(V^*_{n-1}) \times \mathbb{P}(V^*_{n-1})$ *if* $V_n \neq V_n'$.

(2) $Fib^{(3)}(V_{n-1}, V_n, V_n)$ *consists of two irreducible components* A *and* B *with*

$$A = \mathbb{P}(\mathcal{O}_{\mathrm{G}(n-2,V_n)} \oplus \mathcal{U}^*_{\mathrm{G}(n-2,V_n)}(1))\big|_{\mathrm{G}(n-2,V_{n-1})},\ B = Bl_{\Delta}\mathbb{P}(V^*_{n-1}) \times \mathbb{P}(V^*_{n-1}),$$

where A *is the restriction of the projective bundle as in Lemma* 5.10 *over* $\mathrm{G}(n-2, V_{n-1}) \subset \mathrm{G}(n-2, V_n)$*.*

(3) *The intersection* $E_{AB} := A \cap B$ *is given by* $E_{AB} = \mathbb{P}(\mathcal{U}^*_{\mathrm{G}(n-2,V_n)}(1))\big|_{\mathrm{G}(n-2,V_{n-1})} \simeq \mathbb{P}(T_{\mathbb{P}(V^*_{n-1})})$ *in* A*. Also,* E_{AB} *in* B *is the exceptional divisor of* $Bl_\Delta \mathbb{P}(V^*_{n-1}) \times \mathbb{P}(V^*_{n-1})$.

Proof. Part (1) follows from the construction of $F^{(2)} \to \widehat{G}'$.

We show Part (2). The fiber of $F^{(2)} \to \widehat{G}'$ over a point $([V_{n-1}];[V_n],[V_n])$ is $\mathbb{P}(V^*_{n-1}) \times \mathbb{P}(V^*_{n-1})$. The intersection of the fiber $\mathbb{P}(V^*_{n-1}) \times \mathbb{P}(V^*_{n-1})$ with the exceptional locus of $F^{(2)} \to F^{(1)}$ is

$$\{([V_{n-2}],[V_{n-2}];[V_{n-1}];[V_n],[V_n]) \mid V_{n-2} \subset V_{n-1}\} \simeq \mathbb{P}^{n-2},$$

which is nothing but the diagonal of $\mathbb{P}(V^*_{n-1}) \times \mathbb{P}(V^*_{n-1})$. Therefore we have B as an irreducible component of the fiber of $F^{(3)} \to \widehat{G}'$ over the point $([V_{n-1}];[V_n],[V_n])$.

Another component A is a $\mathbb{P}^{n-2}$-bundle over the diagonal of $\mathbb{P}(V^*_{n-1}) \times \mathbb{P}(V^*_{n-1})$ since the exceptional divisor of $F^{(3)} \to F^{(2)}$ is a $\mathbb{P}^{n-2}$-bundle over the exceptional locus of $F^{(2)} \to F^{(1)}$. Since the image on $F^{(1)}$ of the diagonal $\Delta_{V_{n-1}}$ of $\mathbb{P}(V^*_{n-1}) \times \mathbb{P}(V^*_{n-1})$ is equal to $\mathrm{G}(n-2, V_{n-1}) = \mathbb{P}(V^*_{n-1})$ in $\mathrm{G}(n-2, V_n)$, the image of A in $F^{(4)}$ is the restriction of $\mathbb{P}(\mathcal{O}_{\mathrm{G}(n-2,V_n)} \oplus \mathcal{U}^*_{\mathrm{G}(n-2,V_n)}(1))$ over $\mathrm{G}(n-2, V_{n-1})$. Therefore we obtain the description of A as in the statement since $\mathcal{U}^*_{\mathrm{G}(n-2,V_n)}|_{\mathbb{P}(V^*_{n-1})} \simeq T_{\mathbb{P}(V^*_{n-1})}(-1)$ and $\mathcal{N}_{\Delta_{V_{n-1}}} \cong T_{\mathbb{P}(V^*_{n-1})}$ for the normal bundle $\mathcal{N}_{\Delta_{V_{n-1}}}$ of the diagonal $\Delta_{V_{n-1}}$.

It is easy to see the assertion about $A \cap B$. □

Remark 5.12. In [3, Thm. 3.7], the authors studied the relationship between the Hilbert scheme $\mathscr{Y}_0$ of conics in $\mathrm{G}(n-1, V)$ and the stable map compactification of the space of smooth conics in $\mathrm{G}(n-1, V)$, which we denote by $\mathscr{Y}_{\mathrm{st}}$. We interpret this by our study of the birational geometry of $\mathscr{Y}_0$.

By Remark 4.23, $\mathscr{Y}_0 \to \widetilde{\mathscr{Y}}$ is the blow-up along G_σ. By the blow-up $\mathscr{Y}_0 \to \widetilde{\mathscr{Y}}$, the fiber $\rho^{-1}_{\widetilde{\mathscr{Y}}}([V_n])$ becomes the $\mathbb{P}^{n-2}$-bundle $\mathbb{P}(\mathcal{O}_{\mathrm{G}(n-2,V_n)} \oplus \mathcal{U}^*_{\mathrm{G}(n-2,V_n)}(1)) \to \mathrm{G}(n-2, V_n)$ as in Proposition 5.10. Therefore the strict transform Γ of $\rho^{-1}_{\widetilde{\mathscr{Y}}}(G^1_{\mathscr{Y}})$ is a $\mathbb{P}^{n-2}$-bundle to $\mathrm{F}(n-2, n, V)$, where we note that $\mathrm{F}(n-2, n, V)$ is isomorphic to the Hilbert scheme of lines in $\mathrm{G}(n-1, V)$. Let $\widetilde{\mathscr{Y}_0} \to \mathscr{Y}_0$ be the blow-up along Γ. Then $\mathscr{Y}_{\mathrm{st}}$ is obtained by contracting the exceptional divisor over Γ to a $\mathbb{P}^2$-bundle over $\mathrm{F}(n-2, n, V)$.

5.6. **The component A of the fiber $Fib^{(3)}(V_{n-1}, V_n, V_n)$.** Let us fix V_{n-1} and V_n such that $V_{n-1} \subset V_n$ and consider the exceptional set A in the fiber

$$Fib^{(3)}(V_{n-1}, V_n, V_n) \simeq A \cup B \text{ over } ([V_{n-1}];[V_n],[V_n]) \in \widehat{G}'.$$

Since A is $\mathbb{Z}_2$-invariant, this determines the corresponding set $A_{\widetilde{\mathscr{Y}}}$ in the fiber $F_{\widetilde{\mathscr{Y}}} \to G_{\mathscr{Y}}$ over $[V_n]$. We note that $A \simeq \mathbb{P}(\mathcal{O}_{\mathrm{G}(n-2,V_{n-1})} \oplus \mathcal{U}^*_{\mathrm{G}(n-2,V_{n-1})}(1)) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}(V^*_{n-1})} \oplus T_{\mathbb{P}(V^*_{n-1})})$ by Proposition 5.10.

Proposition 5.13. *Define* $A_{\mathscr{Y}_2}$ *to be the strict transform of* $A_{\widetilde{\mathscr{Y}}} \subset \widetilde{\mathscr{Y}}$ *under* $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}}$*, and* $A_{\mathscr{Y}_3}$ *by the image of* $A_{\mathscr{Y}_2}$ *under the morphism* $\mathscr{Y}_2 \to \mathscr{Y}_3$.

(1) *The morphism* $A \to A_{\widetilde{\mathscr{Y}}}$ *contracts the divisor* $E_{AB} = \mathbb{P}(\mathcal{U}^*_{\mathrm{G}(n-2,V_{n-1})}(1))$ *to* $\mathrm{G}(n-3, V_{n-1})$.

(2) *The image* $\mathrm{G}(n-3, V_{n-1})$ *of* E_{AB} *on* $A_{\widetilde{\mathscr{Y}}}$ *is the locus of* σ*-planes. The locus* s_A *of* ρ*-conics in* A *is a section of* $A \to \mathrm{G}(n-2, V_{n-1})$ *corresponding to an injection* $\mathcal{O}_{\mathbb{P}(V^*_{n-1})} \to \mathcal{O}_{\mathbb{P}(V^*_{n-1})} \oplus T_{\mathbb{P}(V^*_{n-1})}$.
(3) $A_{\mathscr{Y}_2} \to A_{\widetilde{\mathscr{Y}}}$ *is the blow-up along the image* $\tilde{s}_A$ *in* $A_{\widetilde{\mathscr{Y}}}$ *of the section* s_A.
(4) *Let* $\widehat{A} := Bl_{s_A}A$ *be the blow-up* $\hat{A}$ *of* A *along the section* s_A*. There exists a natural morphism* $\widehat{A} \to A_{\mathscr{Y}_2}$*, which is the blow-up of* $A_{\mathscr{Y}_2}$ *along the singular locus of* $A_{\mathscr{Y}_2}$.
(5) $A_{\mathscr{Y}_3} \simeq A_{\mathscr{Y}_2}$ *and* $\pi_{A_3} \colon A_{\mathscr{Y}_3} \to \mathrm{G}(n-3, V_{n-1})$ *is a quadric cone fibration, where* $\pi_{A_3} := \pi_{\mathscr{Y}_3}|_{A_{\mathscr{Y}_3}}$.

$$\begin{tikzcd}
 & & \widehat{A} \arrow[dl] \arrow[dr] & \\
A_{\mathscr{Y}_3} \arrow[d, "\pi_{A_3}"'] & A_{\mathscr{Y}_2} \arrow[l, "\simeq"'] \arrow[dr] & & A \arrow[dl] \arrow[d] \\
\mathrm{G}(n-3, V_{n-1}) & & A_{\widetilde{\mathscr{Y}}} & \mathbb{P}(V^*_{n-1})
\end{tikzcd}$$

Proof. (1) follow from Proposition 5.10. (4) is clear and (3) follows once we show (2) since $\mathscr{Y}_2 \to \widetilde{\mathscr{Y}}$ is the blow-up along G_ρ by Proposition 4.22 (1) and $\widetilde{s}_A = G_\rho \cap A_{\widetilde{\mathscr{Y}}}$.

To show (2) and (5), as in the discussion of the subsections 4.6 and 4.7, we first consider the case where $\dim V = 4$ and then use the results to the general cases. In case $\dim V = 4$, $A_{\mathscr{Y}_2} = A_{\widetilde{\mathscr{Y}}}$ is isomorphic to $\mathbb{P}(1^2, 2)$ by Proposition 5.11. Moreover, by Proposition 4.18 (5) (d), the vertex corresponds to a σ-plane and $A_{\widetilde{\mathscr{Y}}} \cap G_\rho$ is a $\mathbb{P}^1$ which is the image of a section of $A \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ associated to an injection $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. Therefore, we also have $A_{\mathscr{Y}_3} \simeq A_{\mathscr{Y}_2} \simeq \mathbb{P}(1^2, 2)$. Now we have finished the proof in case $\dim V = 4$.

We turn to the general cases. First we immediately obtain (5) by the results in case $n = 4$ since $\mathscr{Y}_3 \to \mathrm{G}(n-3, V)$ is the family of $\mathscr{Y}_3 = \mathrm{G}(3, \wedge^2(V/V_{n-3}))$ for 4-dimensional spaces V/V_{n-3}. By comparing the singularities between $A_{\mathscr{Y}_2}$ and $A_{\widetilde{\mathscr{Y}}}$, we see that the image of E_{AB} is the locus of σ-planes. Then the locus s_A of ρ-conics in A is disjoint from E_{AB}. Since s_A is a section of $A \to \mathrm{G}(n-2, V_{n-1})$, s_A corresponds to an injection $\mathcal{O}_{\mathbb{P}(V^*_{n-1})} \to \mathcal{O}_{\mathbb{P}(V^*_{n-1})} \oplus T_{\mathbb{P}(V^*_{n-1})}$.

Finally we show $\mathscr{P}_\rho \cap A_{\mathscr{Y}_3} \simeq \mathbb{P}(\mathfrak{Q}_{V_{n-1}})$. Note that $\mathscr{P}_\rho \cap A_{\mathscr{Y}_3}$ is isomorphic to the exceptional divisor G of $\widehat{A} \to A$, which we determine now. Let $\mathcal{I}_{s_A}$ be the ideal sheaf of the section s_A in A. Note that $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}(V^*_{n-1})} \oplus T_{\mathbb{P}(V^*_{n-1})})}(1)|_{s_A} = \mathcal{O}_{s_A}$. Tensoring $0 \to \mathcal{I}_{s_A} \to \mathcal{O}_A \to \mathcal{O}_{s_A} \to 0$ with $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}(V^*_{n-1})} \oplus T_{\mathbb{P}(V^*_{n-1})})}(1)$ and pushing forward to $\mathbb{P}(V^*_{n-1})$, we see that $\mathcal{I}_{s_A}/\mathcal{I}^2_{s_A} \simeq \Omega_{\mathbb{P}(V^*_{n-1})}$. Therefore G is isomorphic to $\mathbb{P}(T_{\mathbb{P}(V^*_{n-1})})$. Since $\mathbb{P}(T_{\mathbb{P}(V^*_{n-1})})$ is isomorphic to the incident variety $\{([V_{n-3}], [V_{n-2}]) \mid V_{n-3} \subset V_{n-2}\} \subset \mathbb{P}(V_{n-1}) \times \mathbb{P}(V^*_{n-1})$, it follows that $\mathbb{P}(T_{\mathbb{P}(V^*_{n-1})})$ is isomorphic to $\mathbb{P}(T_{\mathbb{P}(V_{n-1})}(-1))$. □

Remark 5.14. Based on Remark 4.19 and Proposition 5.13, we can obtain the following description of $A_{\mathscr{Y}_3} \to \mathrm{G}(n-3, V_{n-1})$, which follows by noting the fiber of $\mathscr{Y}_3 \to \mathrm{G}(n-3, V)$ over $[V_{n-3}]$ is isomorphic to $\mathrm{G}(3, \wedge^2(V/V_{n-3}))$:

Take a point $[V_{n-3}] \in \mathrm{G}(n-3, V_{n-1})$ and let Γ be the fiber of $A_{\mathscr{Y}_3} \to \mathrm{G}(n-3, V_{n-1})$ over $[V_{n-3}]$. The vertex of the quadric cone Γ corresponds to the σ-plane $\mathrm{P}_{V_n/V_{n-3}} = \{\mathbb{C}^2 \subset V_n/V_{n-3}\}$, where we denote by $\mathrm{P}_{V_n/V_{n-3}}$ the σ-plane

in $\mathrm{G}(3,\wedge^2(V/V_{n-3}))$ corresponding to the σ-plane $\mathrm{P}_{V_{n-3}V_n}$. Points $[\mathrm{P}_{V_{n-2}/V_{n-3}}]$ which correspond to ρ-planes and are contained in Γ satisfy $V_{n-3}\subset V_{n-2}$, where we follows the same convention for ρ-planes as for σ-planes. Since Γ is the cone over the Veronese curve $v_2(\mathbb{P}(V_{n-1}/V_{n-3}))$, it is swept out by lines joining $[\mathrm{P}_{V_n/V_{n-3}}]$ and $[\mathrm{P}_{V_{n-2}/V_{n-3}}]$ such that $V_{n-3}\subset V_{n-2}\subset V_{n-1}$.

By this description, we see that $\mathscr{P}_\rho\cap A_{\mathscr{Y}_3}\simeq\mathbb{P}(\mathfrak{Q}_{V_{n-1}})\simeq\mathrm{F}(n-3,n-2,V_{n-1})$, where $\mathfrak{Q}_{V_{n-1}}$ is the universal quotient bundle on $\mathrm{G}(n-3,V_{n-1})$.

Appendix A. **Proof of Proposition 4.9**

Proof of Proposition 4.9. If $\dim a_U\geq n-3$, it is easy to see $\operatorname{rank}\varphi_U\leq 1$ by writing down U using a basis of a_U. This shows one direction of (1).

We show the converse direction of (1). If $\varphi_U=0$, then $\mathbb{P}(U)$ is a plane contained in $\mathrm{G}(n-1,V)$, and hence is a ρ- or σ-plane. Therefore, we see that $\dim a_U\geq n-3$ holds by (4.1). Now we assume that $\operatorname{rank}\varphi_U=1$. Then $q:=\mathscr{Y}_3\cap\mathbb{P}(U)$ is the τ-conic which is the zero locus of φ_U. We will argue depending on the rank of the τ-conic q.

Assume that $\operatorname{rank}q=3$. Note that the dual of the universal subbundle $\mathcal{U}^*$ on $\mathrm{G}(n-1,V)$ restricts as $\mathcal{U}^*|_q\simeq\mathcal{O}(1)^{\oplus 2}_{\mathbb{P}^1}\oplus\mathcal{O}^{\oplus n-3}_{\mathbb{P}^1}$, or $\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}^{\oplus n-2}_{\mathbb{P}^1}$ since $\mathcal{U}^*$ is generated by its global sections and $\deg\mathcal{U}^*|_q=\deg\mathcal{O}_{\mathrm{G}(n-1,V)}(1)|_q=2$ since q is a conic. Let Q be the image of $\mathbb{P}(\mathcal{U}|_q)$ under the natural map $\varphi_{\mathcal{U}}\colon\mathbb{P}(\mathcal{U})\to\mathbb{P}(V)$. Then there are two possibilities; (i) the degree of $\mathbb{P}(\mathcal{U}|_q)\to Q$ is two and Q is a $(n-1)$-plane, i.e., a quadric of rank 1, or (ii) the degree of $\mathbb{P}(\mathcal{U}|_q)\to Q$ is one and Q is a quadric of rank 4 or 3 depending on $\mathcal{U}^*|_q\simeq\mathcal{O}(1)^{\oplus 2}_{\mathbb{P}^1}\oplus\mathcal{O}^{\oplus n-3}_{\mathbb{P}^1}$, or $\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}^{\oplus n-2}_{\mathbb{P}^1}$ respectively. The case (i) is excluded since if Q were a $(n-1)$-plane $\mathbb{P}(V_n)$, then $q\subset\{[U]\in\mathrm{G}(n-1,V)\mid U\subset V_n\}$ and q would be a σ-conic by definition, a contradiction. The case (ii) with $\mathcal{U}^*|_q\simeq\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}^{\oplus n-2}_{\mathbb{P}^1}$ also is excluded since if this happened, then q would be a ρ-conic. Therefore we have the case (ii) with $\mathcal{U}^*|_q\simeq\mathcal{O}(1)^{\oplus 2}_{\mathbb{P}^1}\oplus\mathcal{O}^{\oplus n-3}_{\mathbb{P}^1}$. Then we see that q is a connected family of $(n-1)$-planes in the rank four quadric Q. Since all the rank four quadrics are $\mathrm{SL}(V)$-equivalent, we see that any rank three conic q is also $\mathrm{SL}(V)$-equivalent. Therefore we may assume that q is of the form as in Example 4.4. Then it is easy to see that $a_U=\langle\mathbf{e}_4,\dots,\mathbf{e}_n\rangle$ and hence $\dim a_U=n-3$.

Assume that q is of rank two. Then q is of the form as in Example 4.5. Since q is a τ-conic, $V_{n-2}\neq V'_{n-2}$ and $V_n\neq V'_n$. Then it is easy to see that $a_U=V_{n-2}\cap V'_{n-2}$ and hence $\dim a_U=n-3$.

Finally we assume that q is of rank one. Then the support of q is a line l and l is of the form as in Example 4.5. Let $\mathbf{e}_1,\dots,\mathbf{e}_{n-2}$ be a basis of V_{n-2} and $\mathbf{e}_1,\dots,\mathbf{e}_n$ be a basis of V_n. Then l is spanned by $\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{n-2}\wedge\mathbf{e}_{n-1}$ and $\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{n-2}\wedge\mathbf{e}_n$. Now we pass from $\wedge^{n-1}V$ to $\wedge^2V^*$ and let U' and l' the 3-plane in $\wedge^2V^*$ and the line in $\mathbb{P}(\wedge^2V^*)$. Then l' is spanned by $\mathbf{v}_1:=\mathbf{e}^*_n\wedge\mathbf{e}^*_{n+1}$ and $\mathbf{v}_2:=\mathbf{e}^*_{n-1}\wedge\mathbf{e}^*_{n+1}$. Let $\mathbf{w}:=\sum_{i<j}a_{ij}\mathbf{e}^*_i\wedge\mathbf{e}^*_j$ be a vector such that $\mathbf{v}_1,\mathbf{v}_2,\mathbf{w}$ span U'. Then $\mathrm{G}(2,V^*)\cap\mathbb{P}(U')$ is a rank one conic. Solving the equation

$$(\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2+\mu\mathbf{w})\wedge(\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2+\mu\mathbf{w})=0,$$

we obtain the equation of $\mathrm{G}(2,V^*)\cap\mathbb{P}(U')$. Thus $\mathrm{G}(2,V^*)\cap\mathbb{P}(U')$ is a rank one conic iff $\mathbf{v}_1\wedge\mathbf{w}=\mathbf{v}_2\wedge\mathbf{w}=0$. Therefore we have $\mathbf{w}=a_{n-1n}\mathbf{e}^*_{n-1}\wedge\mathbf{e}^*_n+(\sum_{i\leq n-2}a_{in+1}\mathbf{e}^*_i)\wedge\mathbf{e}^*_{n+1}$. Taking these back to $\wedge^{n-1}V$, we see that U is spanned by $\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{n-2}\wedge\mathbf{e}_{n-1}$ and $\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{n-2}\wedge\mathbf{e}_n$ and $\mathbf{w}=a_{n-1n}\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{n-2}+$

$\sum_{i\leq n-2} a_{in+1}\mathbf{e}_1\wedge\cdots\wedge\check{\mathbf{e}}_i\wedge\ldots\mathbf{e}_n$, where $\check{\mathbf{e}}_i$ means that $\mathbf{e}_i$ is removed. Therefore it is easy to see that a_U is spanned by vectors $\sum b_i\mathbf{e}_i$ with $b_{n-1}=b_n=b_{n+1}=0$ and $\sum(-1)^{n-i}a_{in+1}b_i=0$. Therefore $\dim a_U\geq n-3$. $\square$

Appendix B. The "double spin" coordinates of $\mathrm{G}(3,6)$

In this appendix, we set $V_4=\mathbb{C}^4$ with the standard basis. We can write the irreducible decomposition (4.5) as

$$\wedge^3(\wedge^2V_4)=\Sigma^{(3,1,1,1)}V_4\,\oplus\,\Sigma^{(2,2,2,0)}V_4\simeq\mathsf{S}^2V_4\,\oplus\,\mathsf{S}^2V_4^*,$$

where Σ^β is the Schur functor. We define the projective space $\mathbb{P}(\wedge^3(\wedge^2V_4))=\mathbb{P}(\mathsf{S}^2V_4\,\oplus\,\mathsf{S}^2V_4^*)$. The homogeneous coordinate of $\mathbb{P}(\mathsf{S}^2V_4\,\oplus\,\mathsf{S}^2V_4^*)$ is naturally introduced by $[v_{ij},w_{kl}]$, where v_{ij} and w_{kl} are entries of 4×4 symmetric matrices. Let $\mathcal{I}=\{\{i,j\}\mid 1\leq i<j\leq 4\}$ the index set to write the standard basis of $\wedge^2V_4$, then the homogeneous coordinate of $\mathbb{P}(\wedge^3(\wedge^2V_4))$ is naturally given by the $[p_{IJK}]$ where p_{IJK} is totally anti-symmetric for the indices $I,J,K\in\mathcal{I}$. These two coordinates are related by the above irreducible decomposition. Focusing on the different symmetry properties of the Schur functors, it is rather straightforward to decompose p_{IJK} into the two components. When we use the signature function defined by $\mathbf{e}_{i_1}\wedge\mathbf{e}_{i_2}\wedge\mathbf{e}_{i_3}\wedge\mathbf{e}_{i_4}=\epsilon^{i_1i_2i_3i_4}\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3\wedge\mathbf{e}_4$ for a basis $\mathbf{e}_1,..,\mathbf{e}_4$ of V_4, they are given by

$$v_{ij}=\frac{1}{6}\sum_{k,l,m,n}\epsilon^{klmn}p_{[ik][jl][mn]},\quad w_{kl}=\frac{1}{6}\sum_{a,b,c}\sum_{m,n,q}\epsilon^{kabc}\epsilon^{lmnq}p_{[am][bn][cq]},\tag{B.1}$$

where the square brackets in $p_{[ij][kl][mn]}$ represents the anti-symmetric extensions of the indices, i.e., $p_{[ij][J][K]}=p_{\{ij\}[J][K]}$ for $i<j$ while $p_{[ij][J][K]}=-p_{\{ji\}[J][K]}$ for $i\geq j$. For convenience, we write them in the following (symmetric) matrices:

$$\begin{aligned}v=(v_{ij})&=\begin{pmatrix}2p_{\mathbf{124}} & p_{\mathbf{134}}+p_{\mathbf{125}} & p_{\mathbf{234}}+p_{\mathbf{126}} & p_{\mathbf{146}}-p_{\mathbf{245}}\\ & 2p_{\mathbf{135}} & p_{\mathbf{235}}+p_{\mathbf{136}} & p_{\mathbf{156}}-p_{\mathbf{345}}\\ & & 2p_{\mathbf{236}} & p_{\mathbf{256}}-p_{\mathbf{346}}\\ & & & 2p_{\mathbf{456}}\end{pmatrix},\\ w=(w_{kl})&=\begin{pmatrix}2p_{\mathbf{356}} & -p_{\mathbf{346}}-p_{\mathbf{256}} & p_{\mathbf{345}}+p_{\mathbf{156}} & p_{\mathbf{235}}-p_{\mathbf{136}}\\ & 2p_{\mathbf{246}} & -p_{\mathbf{245}}-p_{\mathbf{146}} & p_{\mathbf{126}}-p_{\mathbf{234}}\\ & & 2p_{\mathbf{145}} & p_{\mathbf{134}}-p_{\mathbf{125}}\\ & & & 2p_{\mathbf{123}}\end{pmatrix},\end{aligned}\tag{B.2}$$

where we ordered the index set $\mathcal{I}$ as $\{\mathbf{1},\mathbf{2},...,\mathbf{6}\}=\{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{3,4\}\}$. Inverting the relations (B.2), we can write the Plücker relations among p_{IJK} in terms of the entries of v and w. After some algebra, we find:

Proposition B.1. *The Plücker ideal I_G of $\mathrm{G}(3,6)\subset\mathbb{P}(\wedge^3(\wedge^2V_4))$ is generated by*

$$\begin{aligned}&|v_{IJ}|-\epsilon_{I\check{I}}\epsilon_{J\check{J}}|w_{\check{I}\check{J}}| \qquad (I,J\in\mathcal{I}),\\ &(v.w)_{ij},\ \ (v.w)_{ii}-(v.w)_{jj}\ \ (i\neq j,1\leq i,j\leq 4),\end{aligned}\tag{B.3}$$

where $\check{I}$ represents the complement of I, i.e., $x\in\mathcal{I}$ such that $x\cup I=\{1,2,3,4\}$ and similarly for $\check{J}$. $|v_{IJ}|$ and $|w_{IJ}|$ represent the 2×2 minors of v and w, respectively, with the rows and columns specified by I and J. $\epsilon_{I\check{I}}$ is the signature of the permutation of the 'ordered' union $I\cup\check{I}$. $(v.w)_{ij}$ is the ij-entry of the matrix multiplication $v.w$.

For all $[v,w] \in V(I_G) \simeq \mathrm{G}(3,6)$, we show the following relations (I.1)-(I.5):

(I.1) $\det\, v = \det\, w$.

By the Laplace expansion of the determinant of 4×4 matrix v, we have $\det\, v = \sum_{J\in\mathcal{I}} \epsilon_{J\check{J}} |v_{IJ}||v_{\check{I}\check{J}}|$. Then, using the first relations of (B.3), we obtain the equality.

(I.2) $v.w = \pm\sqrt{\det\, w}\, id_4$, where id_4 is the 4×4 identity matrix.

Note that the second line of (B.3) may be written in a matrix form $v.w = d\, id_4$ with $d = (v.w)_{11} = \cdots = (v.w)_{44}$. Then, by (I.1), we have $\det v\cdot w = (\det\, w)^2 = d^4$ and hence $d^4 - (\det\, w)^2 = (d^2 - \det\, w)(d^2 + \det\, w) = 0$. We consider a special case; $v = a\, id_4$, $w = a\, id_4$. Then $d = (v.w)_{11} = a^2$. Therefore $d^2 = a^4 = \det\, w$ must holds for all since $V(I_G) \simeq \mathrm{G}(3,6)$ is irreducible. Hence $d = \pm\sqrt{\det\, w}$ as claimed.

(I.3) $\operatorname{rk} w \neq 3$ and also $\operatorname{rk} v \neq 3$.

Assume $\operatorname{rk} w = 3$, then from (I.2) we have $v.w = 0$, which implies $\operatorname{rk} v \leq 1$. However, this contradicts the first relations of (B.3). Hence $\operatorname{rk} w \neq 3$. By symmetry, we also have $\operatorname{rk} v \neq 3$.

(I.4) $\operatorname{rk} w = 2 \Leftrightarrow \operatorname{rk} v = 2$.

When $\operatorname{rk} w = 2$, we see $\operatorname{rk} v \geq 2$ by the first relations of (B.3). From (I.1) and (I.3), we must have $\operatorname{rk} v = 2$. The converse follows in the same way.

(I.5) $\operatorname{rk} w \leq 1 \Leftrightarrow \operatorname{rk} v \leq 1$.

This is immediate from the the first relations of (B.3).

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